

SPECTRAL GAP FOR THE KAC MODEL WITH HARD SPHERE COLLISIONS

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Abstract

We prove the analog of the Kac conjecture for hard sphere collisions.

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1 Introduction

The relation between the microscopic dynamics of a system of N particles, interacting through binary collisions, and the Boltzmann equation — which is supposed to provide a simplified description of this dynamics — remains the source of many challenging problems in mathematics and physics. Since the simplified description of the microscopic dynamics provided by the Boltzmann equation still contains all of the information necessary for describing the hydrodynamics of the system in the appropriate scaling regimes, the Boltzmann equation provides an essential mesoscopic link between the microscopic dynamics and the macroscopic description of the system.

In 1956 Marc Kac introduced a model [6, 7] that contains *only* those features of the microscopic collision process in a dilute gas that are relevant to the derivation of the Boltzmann equation in the large N limit. The original model investigated by Kac involved a caricature of collisions between *Maxwellian molecules*, which means that the force law governing pair collisions is such that the rates at which the various kinematically possible collisions take place depends only on the angle between the in-coming and out-going relative velocities, and not on the magnitude of the relative

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velocity. This affords considerable simplification, and most previous work on the Kac model has been carried out in this context of Maxwellian molecules.

We consider a version of the Kac model where the interaction between the pairs of particles is described by a caricature of hard-sphere collisions, and we solve the *Kac conjecture*, which is explained below, for this more physical variant of his model.

The Kac model describes the evolution of a spatially homogenous “gas” of N particles in terms of a stochastic process that is a random walk, the *Kac walk* on the energy sphere for the N particles. Each binary collision is represented by a jump to a new point on the energy sphere in which only the coordinates of a single pair of particles changes. The collision times arrive in a Poisson stream, where for each pair i, j of particles, the jump rate is a function of $E_{i,j}$, the kinetic energy of the pair of particles. The standard versions of the spatially homogeneous Boltzmann equation are obtained from the Kac walk in the large N limit by choosing the jump rates to be proportional to $E_{i,j}^\gamma$ for some power $0 \leq \gamma \leq 1/2$. The case $\gamma = 0$ is the case of Maxwellian molecules, in which the jump rates are uniform for all pairs, while the case $\gamma = 1/2$ corresponds to hard sphere collisions, in which case the jump rates are not only non-uniform, but are not uniformly bounded from below.

The Kac walk is a reversible process, and the uniform probability measure on the energy sphere is its invariant measure; i.e., its equilibrium measure. The rate at which an initially non-uniform distribution relaxes back to equilibrium under the Kac walk has consequences for the Boltzmann equation if and only if this rate can be controlled uniformly in N .

One way to measure this rate is in terms of a *spectral gap* of the generator of the Kac walk. The Kac conjecture is that this generator has a spectral gap that is bounded below uniformly in N .

As Kac remarked, it is non-trivial even to show that the spectral gap is strictly positive for each fixed N . Up to 2000, the only available lower bounds on the spectral gap were in terms of an inverse power of N [4]. Eventually, Kac’s conjecture was resolved for Maxwellian molecules in [5], which provided no estimate on the gap, and somewhat later in [1, 2] where the gap was computed explicitly.

In the case of hard spheres, the difficulty of proving the Kac conjecture is amplified by the fact that the collision rate for the pair i, j is proportional to $E_{i,j}^{1/2}$. Pairs of slowly moving particles, with a low combined energy, are effectively removed from the collision process. Memory of the initial distribution is only erased in the random collisions, so we need to know that there are not many particles that wait a long time before colliding. The slow, low energy particles are a problem in this regard, and the particles with higher energy must be shown, so to speak, to “make up for this”.

Some results have been obtained for the Kac model with non-Maxwellian collisions. In particular, Villani [9] has studied *entropy production* bounds for the Kac model with non-Maxwellian collisions, though he considers collision rates proportional to $(1 + E_{i,j})^\gamma$, and in this way eliminates the small-energy problem. However, he proves the surprising result that entropy production is bounded below uniformly in N for $\gamma = 1$, the case of “super hard spheres”. While his entropy production bound implies a spectral gap bound for rates proportional to $(1 + E_{i,j})$, it does not seem possible to glean from this any information on uniformity of the spectral gap with rates proportional to $E_{i,j}^{1/2}$, or even $E_{i,j}$.

In the rest of this section, the Kac walk is defined, and we present our main results and outline the strategy of proof.

1.1 The Kac Walk

For $N \in \mathbb{N}$ and $E > 0$, let $\mathcal{S}_{N,E}$ be the set consisting of all vectors $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ with

$$\frac{1}{N} \sum_{j=1}^N v_j^2 = E.$$

A point $v \in \mathcal{S}_{N,E}$ specifies the velocities of a collection of N particles with unit mass. $\mathcal{S}_{N,E}$ is the *energy sphere* for N particles with mean energy E per particle. The Kac walk is a random walk in which each step of the walk corresponds to a binary collision of a single pair of particles. With each collision, the state of the process “jumps” from (v_1, \dots, v_N) to

$$(v_1, v_2, \dots, v_i^*, \dots, v_j^*, \dots, v_N),$$

where only v_i and v_j have changed. Since the process models energy conserving collisions, so that $\mathcal{S}_{N,E}$ may indeed be taken as the state space of the walk, we require that

$$v_i^{*2} + v_j^{*2} = v_i^2 + v_j^2. \quad (1.1)$$

The set of all collisions satisfying (1.1) can be parameterized by

$$v_i^* = v_i \cos \theta + v_j \sin \theta \quad v_j^* = -v_i \sin \theta + v_j \cos \theta. \quad (1.2)$$

We now specify the rate at which these collisions occur. We associate to each pair (i, j) , $i < j$ an exponential random variable $T_{i,j}$ with parameter

$$\lambda_{i,j} = N \binom{N}{2}^{-1} (v_i^2 + v_j^2)^\gamma, \quad (1.3)$$

where $0 \leq \gamma \leq 1$. In more detail, the $T_{i,j}$ are independent, and

$$\Pr\{T_{i,j} > t\} = e^{-t\lambda_{i,j}}.$$

$T_{i,j}$ is the waiting time for particles i and j to collide. The first collision occurs at time

$$T = \min_{i < j} \{T_{i,j}\}. \quad (1.4)$$

At the time T , the pair (i, j) furnishing the minimum collide. Then an angle θ is selected uniformly at random, and the process jumps from (v_1, \dots, v_N) to $(v_1, v_2, \dots, v_i^*, \dots, v_j^*, \dots, v_N)$ with v_i^* and v_j^* given by (1.2). After each collision, the process starts afresh. Let $V(t)$ denote the random state of the process at time t .

It is possible to generalize the model to include a non-uniform rule for selecting the scattering angle θ . Such generalizations are of physical interest, and were investigated in [2]. However, to keep the notation simple, we restrict our attention for most of the paper to the case in which the collision angle is chosen uniformly. This allows us to focus on the much more significant technical difficulties that arise from the consideration of non-uniform jump rates. The case $\gamma = 1/2$ models hard sphere collision and is the case of main physical interest.

The object of our investigation is the spectral gap for the generator of the Markov semigroup associated to the Kac walk. For any continuous function f on $\mathcal{S}_{N,E}$, define the Kac walk generator $L_{N,E}$ by

$$L_{N,E}f(v) = \frac{1}{h} \lim_{h \rightarrow 0} \mathbb{E}\{f(V(h)) - f(v) \mid V(0) = v\}.$$

One readily computes that

$$L_{N,E}f(v) = -N \binom{N}{2}^{-1} \sum_{i < j} (v_i^2 + v_j^2)^\gamma \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(v) - f(R_{i,j,\theta}v)] d\theta \quad (1.5)$$

where

$$(R_{i,j,\theta}v)_k = \begin{cases} v_i^*(\theta) & k = i \\ v_j^*(\theta) & k = j \\ v_k & k \neq i, j \end{cases}.$$

Introducing the notation

$$[f]^{(i,j)}(v) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(R_{i,j,\theta}v) d\theta,$$

we can write the generator more briefly as

$$L_{N,E}f(v) = -N \binom{N}{2}^{-1} \sum_{i < j} (v_i^2 + v_j^2)^\gamma [f(v) - [f]^{(i,j)}(v)].$$

Let \mathcal{H}_E denote the Hilbert space of functions on $\mathcal{S}_{N,E}$ that are square integrable with respect to the uniform probability measure on $\mathcal{S}_{N,E}$. For $f, g \in \mathcal{H}_E$, we denote the inner product of f and g by $\langle f, g \rangle$ and the norm of f by $\|f\|_2$.

Let $\mathcal{E}_{N,E}$ be the Dirichlet form on \mathcal{H} , the Hilbert space given by

$$\mathcal{E}_{N,E}(f, f) = N \binom{N}{2}^{-1} \sum_{i < j} \int_{\mathcal{S}_{N,E}} (v_i^2 + v_j^2)^\gamma (f - [f]^{(i,j)})^2 d\sigma. \quad (1.6)$$

Then

$$\mathcal{E}_{N,E}(f, f) = -\langle f, L_{N,E}f \rangle. \quad (1.7)$$

Evidently, the uniform distribution ($f = 1$) is the unique equilibrium state for this process.

The object of our investigation, the *spectral gap*, is the quantity

$$\Delta_{N,E} := \inf\{\mathcal{E}_{N,E}(f, f) : \|f\|_2 = 1, \langle f, 1 \rangle = 0, f \text{ symmetric}\}, \quad (1.8)$$

where the symmetry in question is symmetry under permutation of coordinates. Note that the subspace of symmetric functions is invariant under $e^{tL_{N,E}}$. It is the spectral gap in this symmetric sector that is relevant to the Boltzmann equation; see [6, 7].

When attempting to determine $\Delta_{N,E}$, a significant difference between the cases $\gamma = 0$ and $\gamma > 0$ lies in the following observation: For each $m \in \mathbb{N}$, the space $\mathcal{P}_{m,N}$ consisting of polynomial functions of v_1, \dots, v_N of degree at most m in each v_j is invariant under $L_{N,E}$, as follows easily from the definition of $L_{N,E}$.

Thus, the eigenfunctions of $L_{N,E}$ will lie in these subspaces, which are finite dimensional. While there is no monotonicity argument that provides an *a priori* guarantee that the eigenfunction furnishing the spectral gap will be a *low degree* polynomial, it is natural to guess that this is the case. Simple computation then show that for all $N \geq 2$,

$$f_0(v) = \sum_{j=1}^N \varphi_0(v_j) \quad \text{where} \quad \varphi_0(w) = w^4 - \frac{3N}{N+2} \quad (1.9)$$

is an eigenfunction of $L_{N,E}$ with eigenvalue

$$\frac{1}{2} \frac{N+2}{N-1} . \quad (1.10)$$

It is natural to guess that (1.10) is the spectral gap Δ_N , and that (1.9) is the gap eigenfunction. (There are a few other symmetric low-degree candidates one might try, but none of them does as well.)

We emphasize that even for $\gamma = 0$, not all of the eigenfunctions of $L_{N,E}$ have this simple structure, though, the gap eigenfunction does, as the proof in [1] shows. It is because of this fact that for $\gamma = 0$, $\lim_{N \rightarrow \infty} \Delta_N$ is exactly equal to the spectral gap of the linearized Boltzmann equation, as we discuss below.

For $\gamma > 0$, there are no polynomial eigenfunctions of $L_{N,E}$, nor are their eigenfunction of the form $\sum_{j=1}^N \varphi(v_j)$. Nonetheless, there are good reasons to expect that the gap eigenfunction for $\gamma > 0$ must be *approximately of this form* for large N , as we explain below.

Therefore, define \mathcal{A}_E to be the subspace of \mathcal{H}_E consisting of functions f such that

$$f(v) = \sum_{j=1}^N \varphi(v_j) ,$$

and define

$$\widehat{\Delta}_{N,E} := \inf \{ \mathcal{E}_{N,E}(f, f) : \|f\|_2 = 1 , \langle f, 1 \rangle = 0 , f \in \mathcal{A}_E \} . \quad (1.11)$$

The subspace \mathcal{A}_E is *not* invariant under $L_{N,E}$, and $\widehat{\Delta}_{N,E}$ is not an eigenvalue of $L_{N,E}$. Nonetheless, if it is indeed true that for large N the gap eigenfunction is *close* to an element of \mathcal{A}_E , one would expect $\widehat{\Delta}_{N,E}$ to be only slightly larger than $\Delta_{N,E}$ for large N .

As we shall show, there is indeed a very close connection between $\widehat{\Delta}_{N,E}$ and $\Delta_{N,E}$, and that solving the more manageable problem of proving a lower bound on the former quantity leads to a lower bound on the latter quantity.

For these reasons, we are interested in how both $\Delta_{N,E}$ and $\widehat{\Delta}_{N,E}$ vary with N and E as γ is held fixed. The dependence on E is a simple matter of scaling.

1.1 LEMMA. *For all $N > 0$, $\gamma \in [0, 1]$, and $E, E' > 0$,*

$$\Delta_{N,E} = \left(\frac{E}{E'} \right)^\gamma \Delta_{N,E'} , \quad (1.12)$$

and the same relation holds for $\widehat{\Delta}_{N,E}$.

Proof. Suppose f is any measurable function on $\mathcal{S}_{N,E'} = S^{N-1}(E')$. Define Sf by

$$Sf(v) = f\left(\sqrt{\frac{E'}{E}}v\right). \quad (1.13)$$

Then, with the uniform probability measure on both spheres, it is clear that $f \mapsto Sf$ is unitary from $L^2(\mathcal{S}_{N,E'})$ to $L^2(\mathcal{S}_{N,E})$, and that

$$N \binom{N}{2}^{-1} \sum_{i < j} \int_{\mathcal{S}_{N,E'}} (v_i^2 + v_j^2)^\gamma (f - [f]^{(i,j)})^2 d\sigma = \left(\frac{E'}{E}\right)^\gamma \mathcal{E}_{N,E}(Sf, Sf). \quad (1.14)$$

□

Note that in the Maxwellian case ($\gamma = 0$), the radius of the sphere is immaterial. On account of this lemma, we simplify our notation: We shall write Δ_N in place of $\Delta_{N,1}$; that is, the spectral gap for N particles with unit energy per particle. Likewise, we shall write \mathcal{E}_N to denote $\mathcal{E}_{N,1}$, \mathcal{H} to denote \mathcal{H}_1 , and \mathcal{A} to denote \mathcal{A}_1 .

The dependence of Δ_N on N is not so simple to determine. The original conjecture of Kac was that for $\gamma = 0$,

$$\liminf_{N \rightarrow \infty} \Delta_N > 0. \quad (1.15)$$

This was proved in [5], with the exact value of Δ_N obtained in [1]. Again for $\gamma = 0$, the Kac conjecture for three-dimensional momentum and energy conserving collisions [7] was proved in [2], and with the exact value of Δ_N in this three-dimensional case in [3].

It is evident that for each N , $\hat{\Delta}_N \geq \Delta_N$, and almost as evident that this inequality is strict. Nonetheless, the weaker conjecture that

$$\liminf_{N \rightarrow \infty} \hat{\Delta}_N > 0 \quad (1.16)$$

will be shown to be a significant stepping-stone towards the proof of (1.15).

1.2 THEOREM. *For all $N \geq 3$,*

$$\hat{\Delta}_N \geq \left(1 - \frac{A_N}{N^2}\right) \hat{\Delta}_{N-1} \quad (1.17)$$

where

$$A_N = \frac{p(N) + \gamma q(N)}{r(N)} \quad (1.18)$$

with

$$\begin{aligned} p(N) &= 5N^7 + 31N^6 + 15N^5 + 131N^4 + 256N^3 - 102N^2 \\ q(N) &= 5N^7 - 5N^6 - 87N^5 - 211N^4 - 164N^3 + 78N^2 \\ r(N) &= (N^2 + 4N - 12)(N - 1)^3(N + 1)(N - 2). \end{aligned} \quad (1.19)$$

Moreover for all N_0 such that $A_N/N^2 < 1$ for all $N \geq N_0$, which is true for all sufficiently large N_0 ,

$$\hat{\Delta}_N \geq N_0^{\gamma-1} \left(\prod_{j=3}^{N_0} \left[1 - \frac{4j+1}{(j-1)^2(j+1)} \right] \right) \prod_{k \geq N_0}^N \left(1 - \frac{A_k}{k^2} \right), \quad (1.20)$$

and in particular, (1.16) is true, with a computable lower bound.

As indicated by (1.17), our proof of (1.16) proceeds by induction on the number of particles, as in our previous works concerning the $\gamma = 0$ case. Toward this end, we note that it is easy to compute Δ_2 .

1.3 LEMMA. *For all $0 \leq \gamma \leq 1$,*

$$\widehat{\Delta}_2 = \Delta_2 = 2^{\gamma+1} .$$

Proof. This is a simple calculation. □

Some remarks on the connection between (1.20) and (1.16) are in order. The estimate (1.20) is only meaningful in case $A_N/N^2 < 1$. However, since $\lim_{N \rightarrow \infty} A_N =: 5(1 + \gamma)$ exists, there exists an N_0 so that $A_N/N^2 < 1$ for all $N \geq N_0$, and for such N_0 ,

$$\prod_{j \geq N_0}^{\infty} \left(1 - \frac{A_N}{N^2}\right) > 0 .$$

Thus, (1.20) reduces the proof of (1.16) to showing that $\widehat{\Delta}_{N_0} > 0$ for *fixed* N_0 . This is relatively easy: By a fairly direct adaptation of the method used in our earlier work on $\gamma = 0$, we prove in Theorem 2.8 that for each $N_0 \geq 3$,

$$\widehat{\Delta}_{N_0} \geq \Delta_{N_0} \geq 4N_0^{\gamma-1} \left(\prod_{j=3}^{N_0} \left[1 - \frac{4j+1}{(j-1)^2(j+1)}\right] \right) > 0 .$$

Altogether, we obtain (1.16).

Moreover, simple computations show that for $\gamma = 1/2$, the “hard sphere” case, $A_N/N^2 \leq 0.542$ for all $N \geq 6$. Hence we may take $N_0 = 6$ in this case. As we shall see, (1.20) leads to good numerical lower bounds on $\widehat{\Delta}_N$.

As we shall see, once Theorem 1.2 has been proved, it is relatively simple to get a lower bound on Δ_N that is uniform in N . The key to this is a decomposition $f = g + h$ of an arbitrary admissible trial function f in the variational principle for Δ_N into pieces $g \in \mathcal{A}_N$ and h that is “harmless”. Thus, we shall prove:

1.4 THEOREM. *For all $N \geq 3$,*

$$\Delta_N \geq \left(1 - \frac{A_N + C_N}{N^2}\right) \Delta_{N-1} \tag{1.21}$$

where A_N is given by (1.18) and where C_N is given by

$$C_N := \sqrt{15} \frac{1-\gamma}{(N-1)^2} N^{5/2} \left[\frac{2}{N-1} + \frac{8N}{(N-2)(N-4)^2} \right]^{1/2} \left(1 - \frac{15}{(N+1)(N+3)}\right)^{-1/2} . \tag{1.22}$$

Moreover, for all N_0 such that $A_N + C_N < N^2$ for all $N > N_0$, which is true for all sufficiently large N_0 ,

$$\Delta_N \geq 4N_0^{\gamma-1} \left(\prod_{j=3}^{N_0} \left[1 - \frac{4j+1}{(j-1)^2(j+1)}\right] \right) \prod_{j \geq N_0}^N \left(1 - \frac{A_N + C_N}{N^2}\right) > 0 .$$

In particular,

$$\liminf_{N \rightarrow \infty} \Delta_N > 0 .$$

1.2 Application to the Kac-Boltzmann equation

Kac's original motivation for introducing the Kac model was to study the *Kac-Boltzmann* equation, which is a caricature of the spatially homogeneous Boltzmann equation for one dimensional velocities:

$$\frac{\partial}{\partial t} f(v, t) = \mathcal{Q}(f, f)(v, t) \quad (1.23)$$

where $f(v, t)$ gives the probability that a randomly selected molecule will have velocity v at time t and where the *collision kernel* $\mathcal{Q}(f, f)$ for the Kac-Boltzmann equation is given by

$$\mathcal{Q}(f, f) = \frac{1}{\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} (v^2 + w^2)^{\gamma} [f(v_*)f(w_*) - f(v)f(w)] d\theta dw . \quad (1.24)$$

with

$$v_* = v \cos \theta + w \sin \theta \quad \text{and} \quad w_* = -v \sin \theta + w \cos \theta . \quad (1.25)$$

Kac proved that there is a close relation between the Kac walk, and the Kac-Boltzmann equation through his notion of *propagation of chaos*. Our next theorem shows another aspect of this close relation. It concerns the *spectral gap of the linearized collision operator*.

To explain this, we first observe that $\mathcal{Q}(f, f) = 0$ if and only if f is a centered Maxwellian density; i.e.,

$$f(v) = \frac{1}{\sqrt{2\pi\Theta}} e^{-v^2/2\Theta} \quad (1.26)$$

for some $\Theta > 0$, where Θ is the second moment of the probability density in (1.26). By an appropriate choice of units, we may suppose that $\Theta = 1$, and we define the *unit Maxwellian density* M by

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \quad (1.27)$$

The *linearized Kac-Boltzmann operator* \mathcal{L} is obtained by considering small perturbations of M of the form

$$f(v) = M(v)[1 + h(v)] \quad (1.28)$$

where h satisfies

$$\int_{\mathbb{R}} h(v) M(v) dv = \int_{\mathbb{R}} v^2 h(v) M(v) dv = 0 , \quad (1.29)$$

and $h(v) \geq -1$ for all v . In this case, (1.28) defines a probability density with unit second moment, as does M . Thinking of h as small, one finds that with f given by (1.28),

$$\mathcal{Q}(f, f) = \frac{1}{\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} (v^2 + w^2)^{\gamma} [h(v_*) + h(w_*) - h(v) - h(w)] M(v) M(w) d\theta dw + \mathcal{O}(h^2) . \quad (1.30)$$

We define

$$\mathcal{L}h(v) = \frac{1}{\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} (v^2 + w^2)^{\gamma} [h(v_*) + h(w_*) - h(v) - h(w)] M(w) d\theta dw , \quad (1.31)$$

then we can rewrite (1.30) as

$$\mathcal{Q}(f, f) = M(v) \mathcal{L}h(v) + \mathcal{O}(h^2) . \quad (1.32)$$

It is easily checked that \mathcal{L} is self adjoint on $L^2(\mathbb{R}, M(v)dv)$. We compute

$$\begin{aligned} \langle h, \mathcal{L}h \rangle_{L^2(\mathbb{R}, M(v)dv)} = \\ \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-\pi}^{\pi} (v^2 + w^2)^{\gamma} [h(v)h(v_*) + h(v)h(w_*) - h^2(v) - h(v)h(w)] M(v)M(w) d\theta dw dv, \end{aligned} \quad (1.33)$$

It is easy to see that the functions $h(v) = 1$ and $h(v) = v^2$ are in the nullspace of \mathcal{L} .

We then define Λ , the *spectral gap* of \mathcal{L} , by

$$\Lambda = \inf \left\{ -\frac{\langle h, \mathcal{L}h \rangle_{L^2(\mathbb{R}, M(v)dv)}}{\|h\|_{L^2(\mathbb{R}, M(v)dv)}^2} : \langle h, 1 \rangle_{L^2(\mathbb{R}, M(v)dv)} = \langle h, v^2 \rangle_{L^2(\mathbb{R}, M(v)dv)} = 0 \right\} \quad (1.34)$$

Our terminology would suggest that the null space of \mathcal{L} is spanned by 1 and v^2 , and that \mathcal{L} is negative semi-definite. This turns out to be the case. It is easy to see this for $\gamma = 0$, since then \mathcal{L} can be expressed as an average over Mehler kernels, and the spectrum of \mathcal{L} computed exactly. For other values of γ , direct computation is not possible.

Our next theorem, together with our analysis of the Kac master equation, shows that not only is \mathcal{L} negative semidefinite, but that Λ is strictly positive, and moreover, provides a computable lower bound on Λ .

1.5 THEOREM. *For all $0 \leq \gamma < 1$,*

$$\Lambda \geq \limsup_{N \rightarrow \infty} \widehat{\Delta}_N. \quad (1.35)$$

If one is mainly interested in lower bounds on Λ , Theorem 1.5 shows that $\widehat{\Delta}_N$ is the main quantity of interest associated to the Kac walk, more so than Δ_N in this particular regard.

1.6 THEOREM. *For $\gamma = 1/2$,*

$$\Lambda \geq 0.0263.$$

Proof. By Theorem 1.5, it suffices to produce a uniform lower bound on $\widehat{\Delta}_N$ for $\gamma = 1/2$. Simple calculations show that $A_6/6^2 \leq 0.54$, and is below this value for all $k > 6$. Thus, we may take $N_0 = 6$ in (1.20). However, experimentation shows that $N_0 = 10$ yields the optimal result. We evaluate

$$\prod_{k=11}^{10^6} (1 - a_k/k^2) = 0.5067\dots$$

numerically, and bound the remainder by an integral comparison. Combining the terms, we obtain $\widehat{\Delta}_N \geq 0.0263$ for all $N \geq 2$. \square

1.7 Remark. If carry out the analogous estimate for $\gamma = 0$, we obtain the bound

$$\Lambda \geq 0.0592$$

in this case. For $\gamma = 0$, the actual value is $\Lambda = 1/2$, as is easily computed and well known, see [2] for discussion. Thus, the lower bound we obtain by this procedure falls short of the actual value by less than one order of magnitude. We may expect the hard sphere bound to be comparably accurate.

The rest of the paper is organized as follows: In Section 2 we set up an induction scheme, and use it to prove a simple estimate on the gap that is uniform in N if the collision rate is proportional to $E_{i,j}^\gamma$ for $\gamma = 0$ and $\gamma = 1$, but only in these cases. The result for $\gamma = 1$ is foreshadowed by the work of Villani [9] on entropy production in the super-hard sphere case, though it is not a consequence of his work. Analysis of this proof highlights the difficulties to be overcome for $0 < \gamma < 1$.

In Section 3 we show how the induction scheme may be adapted to prove a lower bound on $\widehat{\Delta}_N$ that is independent of N . In Section 4, we introduce a decomposition of trial functions to be used in the variational formula for the spectral gap Δ_N , and show how this decomposition reduces the problem of bounding Δ_N to that of bounding $\widehat{\Delta}_N$. In Section 5 we prove Theorem 1.5, and show how Theorem 1.2 leads to an explicit estimate for the linearized Kac-Boltzmann equation. Certain technical results concerning correlations on the sphere, which may have other applications, are proved in Section 6, and a brief Section 7 explain a method for evaluating certain infinite products that we encounter here.

2 The induction

Fix an admissible trial function f in the variational formula for Δ_N . To do the induction, define the *conditional Dirichlet form* $\mathcal{E}_{N,E}(f, f|v_k)$ given by

$$\mathcal{E}_N(f, f|v_k) = (N-1) \binom{N-1}{2}^{-1} \sum_{i < j; i, j \neq k} \int_{S^{N-2}(\sqrt{N-v_k^2})} (v_i^2 + v_j^2)^\gamma (f - [f]^{(i,j)})^2 d\sigma, \quad (2.1)$$

where the integration on the right is only over the “slices” of $S^{N-1}(\sqrt{N})$ at constant values of v_k , so that the result is still a non-trivial function of v_k . Let $d\nu_N(v_k)$ be the marginal distribution induced on $[-\sqrt{N}, \sqrt{N}]$ by the map $v \mapsto v_k$ and the uniform probability measure $d\sigma$ on $S^{N-1}(\sqrt{N})$.

One easily checks that

$$\mathcal{E}_N(f, f) = \frac{N}{N-1} \left(\frac{1}{N} \sum_{k=1}^N \int_{-\sqrt{N}}^{\sqrt{N}} \mathcal{E}_N(f, f|v_k) d\nu_N(v_k) \right). \quad (2.2)$$

Furthermore,

$$\mathcal{E}_N(f, f|v_k) = \mathcal{E}_{N-1, \sqrt{N-v_k^2}}(g, g)$$

where g is the restriction of f to the slice of $S^{N-1}(\sqrt{N})$ at constant v_k . If g were orthogonal to the constants in $L^2(S^{N-2}(\sqrt{N-v_k^2}))$, we could estimate the right hand side in terms of $\Delta_{N-1, \sqrt{N-v_k^2}}$. By Lemma 1.1,

$$\Delta_{N-1, \sqrt{N-v_k^2}} = \left(\frac{N-v_k^2}{N-1} \right)^\gamma \Delta_{N-1}. \quad (2.3)$$

Combining this with (2.2) would yield an estimate for Δ_N in terms of Δ_{N-1} .

However, even if f is orthogonal to the constants in $L^2(\mathcal{S}_{N,E})$, it need not be the case that g is orthogonal to the constants on its slice. To correct for this, we need to add and subtract the average of f over these slices. The average of f over the k th slice is the average of $f(Rv)$ over all rotations that fix the k th axis, which in turn is the orthogonal projection in $L^2(\mathcal{S}_{N,E})$ onto the

subspace of functions depending *only* on the coordinate v_k . For each $k = 1, \dots, N$, we define P_k to be this orthogonal projection. Note that since $P_k f$ depends only on v_k ,

$$\mathcal{E}_N(f, f|v_k) = \mathcal{E}_N(f - P_k f, f - P_k f|v_k) .$$

Then by the definition of the spectral gap and scaling relation (2.3),

$$\mathcal{E}_N(f - P_k f, f - P_k f|v_k) \geq \left(\frac{N - v_k^2}{N - 1} \right)^\gamma \Delta_{N-1} \|f - P_k f\|_{L^2(S^{N-2}(\sqrt{N-v_k^2}))}^2 .$$

Finally, we have the obvious identity

$$\|f - P_k f\|_{L^2(S^{N-2}(\sqrt{N-v_k^2}))}^2 = \|f\|_{L^2(S^{N-2}(\sqrt{N-v_k^2}))}^2 - \|P_k f\|_{L^2(S^{N-2}(\sqrt{N-v_k^2}))}^2 .$$

Altogether, going back to (2.2), one has

$$\mathcal{E}_N(f, f) \geq \frac{N}{N-1} \Delta_{N-1} \left(\frac{1}{N} \sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} \left(\frac{N - v_k^2}{N - 1} \right)^\gamma [f^2 - |P_k f|^2] d\sigma \right) . \quad (2.4)$$

To summarize conveniently our conclusions, we define the quadratic form $\mathcal{F}_N(f, f)$ on $L^2(S^{N-1}(\sqrt{N}))$ as follows:

2.1 DEFINITION.

$$\mathcal{F}_N(f, f) := \frac{1}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} w^{(\gamma)}(v_k) [f - P_k f]^2 d\sigma \right] \quad (2.5)$$

where

$$w^{(\gamma)}(v) = \left(\frac{N - v^2}{N - 1} \right)^\gamma . \quad (2.6)$$

We have proved:

2.2 THEOREM. *For all symmetric $f \in L^2(S^{N-1}(\sqrt{N}))$ with $\|f\|_2^2 = 1$ and with f orthogonal to the constants,*

$$\mathcal{E}_N(f, f) \geq \left[\frac{N}{N-1} \Delta_{N-1} \right] \mathcal{F}_N(f, f) . \quad (2.7)$$

We therefore define

2.3 DEFINITION.

$$\Gamma_N := \inf \{ \mathcal{F}(f, f) : \|f\|_2 = 1 , \langle f, 1 \rangle = 0 , f \text{ symmetric} \} , \quad (2.8)$$

Combining this definition with (1.8), Theorem 2.2 yields

$$\Delta_N \geq \Delta_{N-1} \frac{N}{N-1} \Gamma_N \quad (2.9)$$

If we can succeed in showing that for all $N \geq 3$

$$\Gamma_N \geq \frac{N-1}{N} (1 - a_N) \quad (2.10)$$

where

$$0 \leq a_N \leq 1 \quad \text{for all } N \quad \text{and} \quad \sum_{N=3}^{\infty} a_N < \infty, \quad (2.11)$$

it will follow that for all N ,

$$\Delta_N \geq \Delta_2 \prod_{j=3}^{\infty} (1 - a_j) > 0, \quad (2.12)$$

providing the bound we seek.

2.4 Remark. More significantly, notice that if $f \in \mathcal{A}_N$; i.e., $f(v) = \sum_{j=1}^N \varphi(v_j)$ for some function φ , then the restriction of f to any slice on which v_k is constant belongs to \mathcal{A}_{N-1} . Therefore, if we define $\widehat{\Gamma}_N$ by

$$\widehat{\Gamma}_N := \inf \left\{ \mathcal{F}_N(f, f) \mid \|f\|_2 = 1, \langle f, 1 \rangle = 0, f \in \mathcal{A}_N \right\}, \quad (2.13)$$

we have

$$\widehat{\Delta}_N \geq \widehat{\Delta}_{N-1} \frac{N}{N-1} \widehat{\Gamma}_N, \quad (2.14)$$

and are thus motivated to seek bounds of the type (2.10) and (2.11) for $\widehat{\Gamma}_N$.

We now turn to the task of proving such bounds.

2.1 Some simple but useful bounds

For our first approach to bounding \mathcal{F}_N from below, we first rewrite this quantity as a difference of two terms. Toward this end, we define the self-adjoint operator $P^{(\gamma)}$ by

$$P^{(\gamma)} = \frac{1}{N} \sum_{k=1}^N \left(\frac{N - v_k^2}{N - 1} \right)^{\gamma} P_k. \quad (2.15)$$

For each k , both P_k and the multiplication operator $\left(\frac{N - v_k^2}{N - 1} \right)^{\gamma}$ are commuting and self adjoint, so that $P^{(\gamma)}$ itself is self-adjoint, and even non-negative. Since each P_k is a projection, we have

$$\frac{1}{N} \sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} \left(\frac{N - v_k^2}{N - 1} \right)^{\gamma} |P_k f|^2 d\sigma = \langle f, P^{(\gamma)} f \rangle_{L^2(S^{N-1}(\sqrt{N}))}. \quad (2.16)$$

Define the function $W^{(\gamma)}$ by

$$W^{(\gamma)} = \frac{1}{N} \sum_{k=1}^N w^{(\gamma)}(v_k) = \frac{1}{N} \sum_{k=1}^N \left(\frac{N - v_k^2}{N - 1} \right)^{\gamma}, \quad (2.17)$$

we may rewrite \mathcal{F}_N as

$$\mathcal{F}_N(f, f) = \int_{S^{N-1}(\sqrt{N})} W^{(\gamma)} f^2 d\sigma - \langle f, P^{(\gamma)} f \rangle. \quad (2.18)$$

From an upper bound of $P^{(\gamma)}$, and a lower bound on $W^{(\gamma)}$, we can deduce a lower bound on \mathcal{F}_N . It is easy to deduce a bound on $\langle f, P^{(\gamma)} f \rangle$ for f orthogonal to the constants from the calculation of the spectral gap of the operator $P^{(0)}$ that was made in [1, 2].

2.5 LEMMA. For all N , all $0 \leq \gamma \leq 1$, and all $f \in L^2(S^{N-1}(\sqrt{N}))$ orthogonal to the constants,

$$\langle f, P^\gamma f \rangle \leq \left(\frac{N}{N-1} \right)^\gamma \mu_N \|f\|_2^2 \quad (2.19)$$

where

$$\mu_N = \frac{1}{N} + \frac{3}{N(N+1)} . \quad (2.20)$$

Proof. Using the pointwise upper bound $\left(\frac{N-v_k^2}{N-1} \right) \leq \frac{N}{N-1}$, we have

$$\begin{aligned} \langle f, P^{(\gamma)} f \rangle &= \frac{1}{N} \sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} \left(\frac{N-v_k^2}{N-1} \right)^\gamma |P_k f|^2 d\sigma \leq \\ &\frac{1}{N} \sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} \left(\frac{N}{N-1} \right)^\gamma |P_k f|^2 d\sigma = \left(\frac{N}{N-1} \right)^\gamma \langle f, P^{(0)} f \rangle . \end{aligned} \quad (2.21)$$

$P^{(0)}$ coincides with the operator P analyzed in [1, 2] where it was shown that μ_N is its second largest eigenvalue after the eigenvalue 1 corresponding to the constant function. \square

2.6 LEMMA. For all N , all $0 < \gamma \leq 1$, and for all $v \in S^{N-1}(\sqrt{N})$,

$$\left(\frac{N-1}{N} \right)^{1-\gamma} \leq W^{(\gamma)}(v) \leq 1 . \quad (2.22)$$

Furthermore, for all $v \in S^{N-1}(\sqrt{N})$,

$$W^{(0)}(v) = W^{(1)}(v) = 1 \quad (2.23)$$

Proof. Since $\frac{1}{N} \sum_{k=1}^N v_k^2 = 1$, $\frac{1}{N} \sum_{k=1}^N \left(\frac{N-v_k^2}{N-1} \right) = 1$, Jensen's inequality yields

$$\frac{1}{N} \sum_{k=1}^N \left(\frac{N-v_k^2}{N-1} \right)^\gamma \leq \left(\frac{1}{N} \sum_{k=1}^N \frac{N-v_k^2}{N-1} \right)^\gamma = 1 .$$

To prove the lower bound, note that the minimum of $v \mapsto \frac{1}{N} \sum_{k=1}^N \left(\frac{N-v_k^2}{N-1} \right)^\gamma$ is the same as the minimum of the function

$$(x_1, \dots, x_N) \mapsto \frac{1}{N} \sum_{k=1}^N \left(\frac{N-x_k}{N-1} \right)^\gamma \quad (2.24)$$

on the set of (x_1, \dots, x_N) satisfying

$$x_j \geq 0 \quad \text{for all } j = 1, \dots, N \quad \text{and} \quad \sum_{j=1}^N x_j = N . \quad (2.25)$$

Since the function in (2.24) is concave, the minimum occurs at an extreme point of the domain, and by symmetry, all extreme points yield the same value. Thus the minimum occurs at

$$(x_1, \dots, x_N) = (N, 0, \dots, 0),$$

and the minimum value is $((N-1)/N)^{1-\gamma}$. The assertions about $\gamma = 0$ and $\gamma = 1$ are obvious. \square

2.7 Remark. A simple calculus and convexity argument shows that

$$1 - (1 - \gamma) \frac{1}{N-1} \leq \left(\frac{N-1}{N} \right)^{1-\gamma} \leq 1 - (1 - \gamma) \frac{1}{N} \quad (2.26)$$

for all N and all $0 \leq \gamma \leq 2$. Note also that $\lim_{\gamma \rightarrow 0} \left(\frac{N-1}{N} \right)^{1-\gamma} = \frac{N-1}{N} < 1$ so that although the lower bound in the lemma is sharp, the lack of uniform convergence means that the limiting bound is not sharp.

2.2 Lower bound on Δ_N using the uniform bound on $W^{(\gamma)}$.

We now use the lower bound 2.22 together with Lemma 2.5 to obtain a lower bound on Δ_N . Because 2.22 is only sharp for $\gamma > 0$, and since $\gamma = 0$ has been treated in [1, 2], we only consider $\gamma > 0$ in the next theorem.

2.8 THEOREM. *For all $0 < \gamma \leq 1$, and all $N \geq 2$,*

$$\Delta_N \geq 4N^{\gamma-1} \left(\prod_{j=3}^N \left[1 - \frac{4j+1}{(j-1)^2(j+1)} \right] \right). \quad (2.27)$$

Proof. Fix f orthogonal to the constants. By Lemma 2.5 and 2.6 and (2.18),

$$\begin{aligned} \mathcal{F}_N(f, f) &\geq \left(\frac{N-1}{N} \right)^{1-\gamma} \|f\|_2^2 - \left(\frac{N}{N-1} \right)^\gamma \mu_N \|f\|_2^2 \\ &= \left(\frac{N}{N-1} \right)^{\gamma-2} \left[\frac{N}{N-1} - \left(\frac{N}{N-1} \right)^2 \mu_N \right] \|f\|_2^2 \\ &= \left(\frac{N}{N-1} \right)^{\gamma-2} \left[1 - \frac{4N+1}{(N-1)^2(N+1)} \right] \|f\|_2^2. \end{aligned}$$

Therefore,

$$\Gamma_N \geq \left(\frac{N}{N-1} \right)^{\gamma-1} \left[1 - \frac{4N+1}{(N-1)^2(N+1)} \right].$$

Thus by (2.12),

$$\Delta_N \geq \left(\frac{N}{2} \right)^{\gamma-1} \left(\prod_{j=3}^N \left[1 - \frac{4j+1}{(j-1)^2(j+1)} \right] \right) \Delta_2.$$

\square

By Theorem 7.1,

$$\lim_{N \rightarrow \infty} \prod_{j=3}^N \left[1 - \frac{4j+1}{(j-1)^2(j+1)} \right] = \frac{3}{\Gamma((5+\sqrt{21})/2)\Gamma((5-\sqrt{21})/2)} \approx 0.03881503614 .$$

For $\gamma = 1$, we then have the following result:

2.9 COROLLARY. *For $\gamma = 1$ and all $N \geq 2$, we have*

$$\Delta_N \geq \frac{12N^{\gamma-1}}{\Gamma((5+\sqrt{21})/2)\Gamma((5-\sqrt{21})/2)} \approx 0.1552601446N^{\gamma-1} . \quad (2.28)$$

Using the lower bound from Lemma 2.6 we cannot obtain a lower bound on Δ_N which is uniform in N except when $\gamma = 1$. In the next section we shall obtain a bound on Γ_N , and hence Δ_N , that is much sharper *for large N* that leads to uniform bounds on Δ_N .

2.10 Remark. Nowhere in the proof of Theorem 2.8 have we made any use of the hypothesis that f be symmetric. In fact, the bound proved in Theorem 2.8 is a bound on the spectral gap of $L_{N,1}$ on the whole space, not only the symmetric subspace. In the next section we shall make use of the symmetry hypothesis. In the final section, we explain how it may be avoided, but at the cost of a numerically worse bound.

3 Lower bound on $\hat{\Delta}_N$ and Δ_N for $0 < \gamma < 1$.

We show in this section that Theorem 2.2 can be used as the basis of an inductive approach to bounding $\hat{\Delta}_N$ from below uniformly in N . The principle behind this is that for admissible trial functions in \mathcal{A}_N , i.e., admissible trial functions of the form $f(v) = \sum_{j=1}^N \varphi(v_j)$, the probability density f^2 cannot be too concentrated in places where the weight $W^{(\gamma)}$ is significantly less than 1, and thus one can improve upon the uniform lower bound on $W^{(\gamma)}$. However, our actual proof makes somewhat indirect use of this.

The key to taking advantage of the special form of trial functions $f \in \mathcal{A}_N$ is provided by certain correlation inequalities.

3.1 The correlation operators

Many of the estimates we shall make to bound $\hat{\Delta}_N$ would be trivial if it were the case that the functions $\varphi(v_j)$, $j = 1, \dots, N$ were independent random variables for the uniform distribution on the energy sphere.

For large N , the different velocities v_j , are, in fact, nearly independent. This is due to the fact that the unit Gaussian distribution

$$d\gamma_N := (2\pi)^{-N/2} e^{-|v|^2/2} d^N v$$

is very tightly concentrated on a close neighborhood of $S^{N-1}(\sqrt{N})$, and in this sense is very close to the uniform measure on $S^{N-1}(\sqrt{N})$, $d\sigma$. This is an instance of the *equivalence of ensembles* in statistical mechanics — in this case $d\sigma$ is the *microcanonical ensemble* for a gas of N free particles

with unit mean energy per particle, and $d\gamma_N$ is the *canonical ensemble* for the same system. Under the canonical ensemble, the various different velocities *are* statistically independent.

We shall require a quantitative measure of the amount of dependence that there is for each finite N . We do this through the *correlation operators*, the first and simplest of which we now define.

3.1 DEFINITION (The pair correlation operator K). For any unit vector u on \mathbb{R}^N , the map $v \mapsto v \cdot u$ maps $S^{N-1}(\sqrt{N})$ onto $[-\sqrt{N}, \sqrt{N}]$, and pushes the uniform probability $d\sigma$ forward onto a probability measure $d\nu_N$ on $[-\sqrt{N}, \sqrt{N}]$ that is independent of u . We define an operator K on $L^2([-1, 1], d\nu_N)$ by

$$\begin{aligned} \int_{S^{N-1}(\sqrt{N})} \varphi(v_1) \psi(v_2) d\sigma &= \int_{S^{N-1}(\sqrt{N})} \varphi(v_1) (K\psi)(v_1) d\sigma \\ &= \int_{-\sqrt{N}}^{\sqrt{N}} \varphi(w) K\varphi(w) d\nu_N(w) . \end{aligned} \quad (3.1)$$

In [2] the eigenvalues α_k of this operator have been computed. The eigenfunctions are polynomials of degree k . All of the odd polynomials are in the kernel of K . Moreover

$$|\alpha_k| > |\alpha_{k+2}| \quad (3.2)$$

and

$$\alpha_0 = 1, \quad \alpha_2 = -\frac{1}{N-1}, \quad \alpha_4 = \frac{3}{(N-1)(N+1)}, \quad \alpha_6 = -\frac{15}{(N-1)(N+1)(N+3)}. \quad (3.3)$$

For each even k , the eigenspace for the eigenvalue α_k is one-dimensional, and is spanned by an even polynomial of degree k .

Now let φ and ψ be functions on $[-\sqrt{N}, \sqrt{N}]$ that are orthogonal to the constants; i.e., the eigenspace of α_0 . Then, by the definition above, and what we have said about the eigenvalues of K , since $|\alpha_2| = 1/(N-1)$,

$$\begin{aligned} \left| \int \varphi(v_1) \psi(v_2) d\sigma - \left(\int \varphi(v_1) d\sigma \right) \left(\int \psi(v_2) d\sigma \right) \right| &= \left| \int_{-\sqrt{N}}^{\sqrt{N}} \varphi(w) K\varphi(w) d\nu_N(w) \right| \\ &\leq \frac{1}{N-1} \|\varphi\|_2 \|\psi\|_2 . \end{aligned}$$

where $\|\varphi\|_2^2$ denotes $\int_{S^{N-1}(\sqrt{N})} |\varphi(v_1)|^2 d\sigma = \int_{[-\sqrt{N}, \sqrt{N}]} |\varphi(w)|^2 d\nu_N(w)$. Thus for large N , the random variables $\varphi(v_1)$ and $\psi(v_2)$ are almost uncorrelated.

If we know that $\varphi(v_1)$ and $\psi(v_2)$ are not only orthogonal to the constants but are also orthogonal to v_1^2 and v_2^2 respectively, then φ and ψ are orthogonal to the α_2 eigenspace as well as the α_0 eigenspace, and we obtain the stronger bound

$$\left| \int \varphi(v_1) \psi(v_2) d\sigma - \left(\int \varphi(v_1) d\sigma \right) \left(\int \psi(v_2) d\sigma \right) \right| \leq \frac{3}{N^2-1} \|\varphi\|_2 \|\psi\|_2 .$$

In this case, we get a much stronger bound on correlations. The following lemma will make this stronger bound available to us and we shall use it a number of times in what follows.

3.2 LEMMA. *Let $g \in L^2(S^{N-1}(\sqrt{N}))$ be orthogonal to the constants, and in \mathcal{A}_N , i.e., of the form $\sum_{k=1}^N \varphi(v_k)$. The choice of φ is not unique, but among the possible choices, there is always one with the property that for each k , $\varphi(v_k)$ is orthogonal to both 1 and v_k^2 in $L^2(S^{N-1}(\sqrt{N}))$.*

Proof. Since $\int_{S^{N-1}(\sqrt{N})} \varphi(v_k) d\sigma$ is independent of k , we have that for each k ,

$$0 = \int_{S^{N-1}(\sqrt{N})} g(v) d\sigma = N \int_{S^{N-1}(\sqrt{N})} \varphi(v_k) d\sigma ,$$

and so $\int_{S^{N-1}(\sqrt{N})} \varphi(v_k) d\sigma = 0$. Next, let $\eta(v_k) := v_k^2 - 1$ and define

$$\tilde{\varphi}(v_k) = \varphi(v_k) - \left[\left(\int_{S^{N-1}(\sqrt{N})} \eta^2(v_k) d\sigma \right)^{-2} \int_{S^{N-1}(\sqrt{N})} \varphi(v_k) \eta(v_k) d\sigma \right] \eta(v_k) .$$

By symmetry, the coefficient of $\eta(v_k)$ does not depend on k , and then since $\sum_{k=1}^N \eta(v_k) = 0$, it follows that

$$\sum_{k=1}^N \tilde{\varphi}(v_k) = \sum_{k=1}^N \varphi(v_k) = g(v) .$$

By construction, $\tilde{\varphi}(v_k)$ is orthogonal to both 1 and v_k^2 in $L^2(S^{N-1}(\sqrt{N}))$. □

Thus, we may assume henceforth that

$$\int_{S^{N-1}(\sqrt{N})} \varphi(v_1) d\sigma = 0 , \quad \text{and} \quad \int_{S^{N-1}(\sqrt{N})} \varphi(v_1) v_1^2 d\sigma = 0 . \quad (3.4)$$

The orthogonality provided by Lemma 3.2 has consequences that are summarized in the next lemma, which shall be used several times in what follows.

3.3 LEMMA. *Let $g \in L^2(S^{N-1}(\sqrt{N}))$ be in \mathcal{A}_N where for each k , $\varphi(v_k)$ is orthogonal to both 1 and v_k^2 in $L^2(S^{N-1}(\sqrt{N}))$. Then, for each k ,*

$$N \left(1 - \frac{15}{(N+1)(N+3)} \right) \|\varphi(v_k)\|_2^2 \leq \|g\|_2^2 \leq N \left(1 + \frac{3}{N+1} \right) \|\varphi(v_k)\|_2^2 . \quad (3.5)$$

Proof. Since

$$\|g\|^2 = N\|\varphi\|^2 + N(N-1) \int_{S^{N-1}(\sqrt{N})} \varphi(v_1) (K\varphi)(v_1) d\sigma$$

we find on account of (3.4) and the eigenvalues of the K operator listed in (3.3) that α_4 is relevant for the upper bound, and α_6 for the lower bound. □

We shall also make use of higher-order correlation operators, a whole family of which is studied in Section 4. For $N \geq 4$, the operator $K_{N,2}$, acting on functions ψ on the disk of radius \sqrt{N} in \mathbb{R}^2 , is defined through the quadratic form

$$\int_{S^{N-1}(\sqrt{N})} \psi(v_1, v_2) [K_{N,2}\psi](v_1, v_2) d\sigma = \int_{S^{N-1}(\sqrt{N})} \psi(v_1, v_2) \psi(v_{N-1}, v_N) d\sigma ,$$

in analogy with (3.1). (For $N = 3$, $K_{3,2}$ is defined in terms of projection of functions depending only on v_2 and v_3 onto the subspace of functions depending only on v_1 .)

However, most of what follows depends on the properties of the single-particle correlation operator K , and we postpone further analysis of $K_{N,2}$ to Section 4, from which we shall quote results as needed.

We close this subsection with a few comments on how correlations bounds may be used to show that for admissible trial function $f \in \mathcal{A}_N$, f^2 must be largely concentrated on configurations v for which $W^{(\gamma)}$ is very close to 1.

Suppose $f(v) = \sum_{k=1}^N \varphi(v_k)$ is an admissible trial function, with $\varphi(v_k)$ orthogonal to 1 and v_k^2 in $L^2(S^{N-1}(\sqrt{N}))$. Then

$$1 = \int_{S^{N-1}(\sqrt{N})} f^2 d\sigma = \sum_{j,k=1}^N \int_{S^{N-1}(\sqrt{N})} \varphi(v_j) \varphi(v_k) d\sigma .$$

If we make the assumption that the $\varphi(v_k)$ are exactly independent, all terms with $j \neq k$ vanish, and we have that

$$\sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} \varphi^2(v_k) d\sigma = 1 .$$

Again assuming that the coordinate functions are *exactly* independent,

$$N = \int_{S^{N-1}(\sqrt{N})} N f^2 d\sigma = \sum_{j,k,\ell=1}^N \int_{S^{N-1}(\sqrt{N})} v_j^2 \varphi(v_k) \varphi(v_\ell) d\sigma = \sum_{j,k=1}^N \int_{S^{N-1}(\sqrt{N})} v_j^2 \varphi(v_k)^2 d\sigma .$$

This reduces to $\sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} v_k^2 \varphi(v_k)^2 d\sigma = 1$. A similar calculation using $N^2 = \sum_{i,j=1}^N v_i^2 v_j^2$ leads

to $\sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} v_k^4 \varphi(v_k)^2 d\sigma = 2N$. From here, making further use of the assumed independence, one readily derives

$$\int_{S^{N-1}(\sqrt{N})} \left[\sum_{k=1}^N v_k^4 \right] f^2 d\sigma \leq 2N + 3(N-1) \leq 5N \quad (3.6)$$

for all N .

Recall that $W^{(\gamma)}$ has its minimum at the points of $S^{N-1}(\sqrt{N})$ at which $\sum_{k=1}^N v_k^4 = N^2$. In fact, assuming the bound (3.6), it is not hard to show, using Chebychev's inequality, that

$$\int_{S^{N-1}(\sqrt{N})} W^{(\gamma)} f^2 d\sigma \geq 1 - \frac{C}{N^{3/2}}$$

for a computable constant C . This would resolve the main difficulty we encountered in the previous section. In our actual proof, we employ a somewhat more intricate argument that gives us $\mathcal{O}(1/N^2)$ errors, but we hope this heuristic discussion has explained the utility of the correlation bounds we investigate next.

3.2 Lower bound on $\widehat{\Delta}_N$.

We now lay the groundwork for the proof of Theorem 1.2. We introduce a second approach to bounding $\mathcal{F}_N(f, f)$ from below that will yield more incisive bounds for large values of N . The starting point for this approach uses the original formula for $\mathcal{F}_N(f, f)$ from Definition 2.1:

$$\mathcal{F}_N(f, f) = \frac{1}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} w^{(\gamma)}(v_k) [f - P_k f]^2 d\sigma \right]$$

where $w^{(\gamma)}(v) = \left(\frac{N - v^2}{N - 1} \right)^\gamma$. Note that the integrand is positive, and we will exploit some of the cancelations between f and $P_k f$.

3.4 LEMMA. *Let f have the form $f = \sum_{j=1}^N \varphi(v_j)$ with φ orthogonal to 1 and v^2 . Then for all $N \geq 3$,*

$$\mathcal{F}_N(f, f) \geq \frac{N-1}{N} \left(1 - \frac{A_N}{N^2} \right) \|f\|_2^2 \quad (3.7)$$

where A_N is given by (1.18), and where $p(N)$, $q(N)$ and $r(N)$ are given by (1.19).

Theorem 1.2 is an immediate consequence of this Lemma, and previous observation:

Proof of Theorem 1.2. It follows from the definition of $\widehat{\Gamma}_N$ in (2.13) and Lemma 3.4 that $\widehat{\Gamma}_N \geq \frac{N-1}{N} \left(1 - \frac{A_N}{N^2} \right)$. It then follows from Remark 2.4 that the bound in Theorem 1.2 is valid. \square

To get a close estimate on $\mathcal{F}_N(f, f)$, we need to do a number of exact calculations that can be done with polynomials. The following lemma will give us the reduction from $w^{(\gamma)}(v)$ to a polynomial weight function.

3.5 LEMMA. *For all $0 < \gamma < 1$ and all $x > -1$,*

$$(1+x)^\gamma \geq 1 + \gamma x - (1-\gamma)x^2. \quad (3.8)$$

Furthermore, for all γ such that

$$\left(\frac{\gamma}{2} \right)^{1/(2-\gamma)} \geq \frac{1-\gamma}{2-\gamma}, \quad (3.9)$$

the function

$$x \mapsto (1-\gamma)x^2 + (1+x)^\gamma \quad (3.10)$$

is strictly monotone increasing on $(-1, \infty)$.

Proof. Let $\eta(x)$ be defined by $\eta(x) := (1+x)^\gamma - [1 + \gamma x - (1-\gamma)x^2]$. Note that

$$\eta''(x) = (1-\gamma)(2 - \gamma(1+x)^{\gamma-2}).$$

Thus, $\eta''(x) = 0$ has the single solution $x = x_*$ where

$$x_* := (\gamma/2)^{1/(2-\gamma)} - 1.$$

Note that η is convex on (x_*, ∞) , and concave on $(-1, x_*]$, and also that $-1 < x_* < 0$.

Since η is concave on $[-1, x_*]$,

$$\min\{\eta(x) : -1 \leq x \leq x_*\} = \min\{\eta(-1), \eta(x_*)\} = \eta(-1) = 0.$$

Since η is convex on (x_*, ∞) , and this interval contains a point, namely 0, at which η' vanishes, the minimum of η over this interval is attained at $x = 0$, and thus η is non-negative on (x_*, ∞) as well as on $[-1, x_*]$.

For the second part, define $\xi(x) := (1 - \gamma)x^2 + (1 + x)^\gamma$, and note that $\xi''(x) = \eta''(x)$, so that with x_* defined as above, $\xi''(x_*) = 0$. Direct computation shows that $\xi'''(x) > 0$ on $(-1, \infty)$, and so ξ' is a strictly convex function on $(-1, \infty)$. It is therefore minimized at x_* . Computing $\xi'(x_*)$, one finds

$$\xi'(x_*) = 2 \left[(2 - \gamma) \left(\frac{\gamma}{2} \right)^{1/(2-\gamma)} - (1 - \gamma) \right].$$

This is positive if and only if (3.9) is satisfied. Thus, ξ' is strictly positive if and only if (3.9) is satisfied. \square

Lemma 3.5 gives us the lower bound

$$w^{(\gamma)}(v) \geq m(v) := 1 + \gamma \left(\frac{1 - v^2}{N - 1} \right) - (1 - \gamma) \left(\frac{1 - v^2}{N - 1} \right)^2.$$

Note that

$$m(v) = \left[1 - (1 - \gamma) \left(\frac{1 - v^2}{N - 1} \right) \right] \left[\left(\frac{1 - v^2}{N - 1} \right) + 1 \right] > 0.$$

Then

$$\mathcal{F}_N(f, f) \geq \frac{1}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} m(v_k) [f - P_k f]^2 d\sigma \right] := \mathcal{G}_N(f, f),$$

and so it suffices to prove (3.7) with $\mathcal{G}_N(f, f)$ in place of $\mathcal{F}_N(f, f)$.

Proof of Lemma 3.4. Since $f = \sum_{j=1}^N \varphi(v_j)$,

$$P_k f(v) = \varphi(v_k) + (N - 1)K\varphi(v_k) \quad \text{and hence} \quad f(v) - P_k f(v) = \sum_{j \neq k} \varphi(v_j) - (N - 1)K\varphi(v_k),$$

where we have used the K operator defined in (3.1). Developing the square yields

$$\begin{aligned} \mathcal{G}_N(f, f) &= \frac{1}{N} \sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} m(v_k) \left[\sum_{j \neq k} \varphi(v_j) - (N - 1)K\varphi(v_k) \right]^2 d\sigma \\ &= (N - 1) \int_{S^{N-1}(\sqrt{N})} m(v_1) \varphi^2(v_2) d\sigma \\ &+ (N - 1)(N - 2) \int_{S^{N-1}(\sqrt{N})} m(v_1) \varphi(v_2) \varphi(v_3) d\sigma \\ &- 2(N - 1)^2 \int_{S^{N-1}(\sqrt{N})} m(v_1) K\varphi(v_1) \varphi(v_2) d\sigma \\ &+ (N - 1)^2 \int_{S^{N-1}(\sqrt{N})} m(v_1) (K\varphi(v_2))^2 d\sigma \end{aligned} \tag{3.11}$$

Since

$$\int_{S^{N-1}(\sqrt{N})} m(v_1) K \varphi(v_1) \varphi(v_2) d\sigma = \int_{S^{N-1}(\sqrt{N})} m(v_1) K \varphi(v_1) K \varphi(v_1) d\sigma ,$$

and because

$$\int_{S^{N-1}(\sqrt{N})} m(v_1) \varphi^2(v_2) d\sigma = \int_{S^{N-1}(\sqrt{N})} K m(v_1) \varphi^2(v_1) d\sigma ,$$

$$\begin{aligned} \mathcal{G}_N(f, f) &= (N-1) \int_{S^{N-1}(\sqrt{N})} K m(v_1) \varphi^2(v_1) d\sigma \\ &+ (N-1)(N-2) \int_{S^{N-1}(\sqrt{N})} m(v_1) \varphi(v_2) \varphi(v_3) d\sigma \\ &- (N-1)^2 \int_{S^{N-1}(\sqrt{N})} m(v_1) (K \varphi(v_1))^2 d\sigma \\ &= I_1 + I_2 + I_3 . \end{aligned} \quad (3.12)$$

Of the three integrals, I_3 is the easiest to estimate. Noting that $m(v) \leq 1 + \gamma/(N-1)$,

$$\begin{aligned} I_3 &\geq -(N-1)^2 \left(1 + \frac{\gamma}{N-1}\right) \|K \varphi\|^2 \\ &\geq -(1 + \frac{\gamma}{N-1}) \frac{9}{(N+1)^2} \|\varphi\|^2 , \end{aligned}$$

and using Lemma 3.3 we find the lower bound

$$I_3 \geq - \left(1 + \frac{\gamma}{N-1}\right) \left(1 - \frac{15}{(N+1)(N+3)}\right)^{-1} \frac{9}{(N+1)^2 N} \|f\|_2^2 . \quad (3.13)$$

Next, we estimate I_1 . Since,

$$m(v) = \left[1 + \frac{\gamma}{N-1} - \frac{1-\gamma}{(N-1)^2}\right] - \left[\frac{\gamma}{N-1} - 2\frac{1-\gamma}{(N-1)^2}\right] v^2 - \frac{1-\gamma}{(N-1)^2} v^4 ,$$

and because

$$K v^2 = -\frac{1}{N-1} (v^2 - N) \quad \text{and} \quad K v^4 = \frac{3}{N^2-1} (v^2 - N)^2 ,$$

$$\begin{aligned} K m(v) &= \left[1 + \frac{\gamma}{N-1} - \frac{1-\gamma}{(N-1)^2}\right] \\ &+ \left[\frac{\gamma}{N-1} - 2\frac{1-\gamma}{(N-1)^2}\right] \frac{1}{N-1} (v^2 - N) - \frac{1-\gamma}{(N-1)^2} \frac{3}{N^2-1} (v^2 - N)^2 . \end{aligned} \quad (3.14)$$

Introducing $x = (v^2 - N)/(N-1)$, so that $-N/(N-1) \leq x \leq 0$, we have

$$K m(v) = \left[1 + \frac{\gamma}{N-1} - \frac{1-\gamma}{(N-1)^2}\right] + \left[\frac{\gamma}{N-1} - 2\frac{1-\gamma}{(N-1)^2}\right] x - \frac{3(1-\gamma)}{N^2-1} x^2 .$$

The right hand side is a concave function of x , so the minimum occurs at either $x = 0$ or $x = -N/(N-1)$. Direct computation shows that the minimum occurs at $x = -N/(N-1)$, and making a few simplifying estimates, we obtain the bound

$$Km(v) \geq 1 - \frac{2-\gamma}{(N-1)^2} . \quad (3.15)$$

In fact, not making the simplifying assumptions, we have the stronger bound

$$Km(v) \geq 1 - \frac{2-\gamma}{(N-1)^2} + \frac{(1-\gamma)(2N-1)}{(N-1)^3(N+1)} . \quad (3.16)$$

Thus, using (3.15),

$$\begin{aligned} I_1 &\geq (N-1) \left(1 - \frac{2-\gamma}{(N-1)^2}\right) \int_{S^{N-1}(\sqrt{N})} \varphi^2(v_1) d\sigma \\ &\geq \frac{N-1}{N} \left(1 - \frac{2-\gamma}{(N-1)^2}\right) \left[\|f\|_2^2 - N(N-1) \int_{S^{N-1}(\sqrt{N})} \varphi(v_2)\varphi(v_3) d\sigma \right] . \end{aligned} \quad (3.17)$$

Adding the right side of (3.17) to I_2 , we obtain

$$\begin{aligned} I_1 + I_2 &\geq \frac{N-1}{N} \left(1 - \frac{2-\gamma}{(N-1)^2}\right) \|f\|_2^2 \\ &\quad + (N-1) \int_{S^{N-1}(\sqrt{N})} \left[(N-2)m(v_1) - \left(1 - \frac{2-\gamma}{(N-1)^2}\right) (N-1) \right] \varphi(v_2)\varphi(v_3) d\sigma \\ &=: \frac{N-1}{N} \left(1 - \frac{2-\gamma}{(N-1)^2}\right) \|f\|_2^2 + J \end{aligned} \quad (3.18)$$

where the last line defines J . We compute

$$\begin{aligned} \left[(N-2)m(v_1) - \left(1 - \frac{2-\gamma}{(N-1)^2}\right) (N-1) \right] &= -(1-\gamma) \left(1 - \frac{N}{(N-1)^2}\right) \\ &\quad - (N-2) \left[\frac{\gamma}{N-1} - 2\frac{1-\gamma}{(N-1)^2} \right] v_1^2 \\ &\quad - (N-2) \frac{1-\gamma}{(N-1)^2} v_1^4 \end{aligned} \quad (3.19)$$

Therefore,

$$\begin{aligned} J &= -(1-\gamma) \left(1 - \frac{N}{(N-1)^2}\right) (N-1) (\varphi, K\varphi) \\ &\quad - (N-2) \left[\gamma - 2\frac{1-\gamma}{(N-1)} \right] \int_{S^{N-1}(\sqrt{N})} v_1^2 \varphi(v_2)\varphi(v_3) d\sigma \\ &\quad - (N-2) \frac{1-\gamma}{(N-1)} \int_{S^{N-1}(\sqrt{N})} v_1^4 \varphi(v_2)\varphi(v_3) d\sigma . \end{aligned}$$

To estimate the moments we use the following lemma:

3.6 LEMMA.

$$\begin{aligned} \int_{S^{N-1}(\sqrt{N})} v_1^2 \varphi(v_2)\varphi(v_3) d\sigma &= \frac{([N-2v^2]\varphi, K\varphi)}{N-2} \\ \int_{S^{N-1}(\sqrt{N})} v_1^4 \varphi(v_2)\varphi(v_3) d\sigma &= \frac{([N^2-4Nv^2+2v^4]\varphi, K\varphi) + 2(v^2\varphi, K(v^2\varphi))}{(N-2)^2} \end{aligned} \quad (3.20)$$

where (ψ, φ) denotes the inner product

$$\int_{-\sqrt{N}}^{\sqrt{N}} \psi(v) \varphi(v) d\nu_N = \int_{S^{N-1}(\sqrt{N})} \psi(v_1) \varphi(v_1) d\sigma, \quad (3.21)$$

and $\|\varphi\|^2 = (\varphi, \varphi)$.

Proof. Let $\psi_j(v) = v_1^{2j}$. Letting $P_{\{2,3\}}$ denote the orthogonal projection in \mathcal{H} onto the subspace of functions depending only on v_2 and v_3 . Then

$$\int_{S^{N-1}(\sqrt{N})} v_1^{2j} \varphi(v_2) \varphi(v_3) d\sigma = \int_{S^{N-1}(\sqrt{N})} P_{\{2,3\}} \psi_j(v_2, v_3) \varphi(v_2) \varphi(v_3) d\sigma.$$

It is easy to compute $P_{\{2,3\}} \psi_j$ using formulas for the operator $K_{N,2}$ deduced in the final section. In Lemma 6.4, it is shown that

$$P_{\{2,3\}} \psi_2(v_2, v_3) = \frac{1}{N-3} [N - (v_2^2 + v_3^2)] \quad (3.22)$$

and

$$P_{\{2,3\}} \psi_4(v_2, v_3) = \frac{3}{(N-3)(N-1)} [N - (v_2^2 + v_3^2)]^2. \quad (3.23)$$

From here, the proof is a simple calculation using the definition of the K operator. \square

Returning to the proof of Lemma 3.4, we have

$$\begin{aligned} J &\geq -(1-\gamma) \left(1 - \frac{N}{(N-1)^2} \right) (N-1) (\varphi, K\varphi) \\ &\quad - \left[\gamma - 2 \frac{1-\gamma}{(N-1)} \right] ([N-2v^2]\varphi, K\varphi) \\ &\quad - (1-\gamma) \frac{([N^2-4Nv^2+2v^4]\varphi, K\varphi) + 2(v^2\varphi, K(v^2\varphi))}{(N-1)(N-2)} \\ &=: B_1(\varphi, K\varphi) + B_2(v^2\varphi, K\varphi) + B_3[(v^4\varphi, K\varphi) + (v^2\varphi, Kv^2\varphi)]. \end{aligned}$$

Collecting terms, we find

$$B_1 = -(N-1) \left[1 + \frac{(2+\gamma)N^2 - (5\gamma+2)N + 2}{(N-1)^2(N-2)} \right].$$

Simple computations show that the quantity in square brackets is positive for all $0 \leq \gamma \leq 1$ and $N \geq 3$.

Likewise,

$$B_2 = - \left[2\gamma + \frac{8(1-\gamma)}{(N-1)(N-2)} \right] \quad \text{and} \quad B_3 = - \frac{2(1-\gamma)}{(N-1)(N-2)}.$$

By Schwarz's inequality, $|(v^4\varphi, K\varphi)| \leq \|v^4\varphi\| \|K\varphi\| \leq \frac{3N^2}{N^2-1} \|\varphi\|^2$, and likewise,

$$(v^2\varphi, Kv^2\varphi) \leq \frac{3N^2}{N^2-1} \|\varphi\|^2.$$

In both cases we have used the fact that φ as well as $v^2\varphi$ are orthogonal to the constant function. In a similar fashion we find that $|(v^2\varphi, K\varphi)| \leq \frac{3N}{N^2-1}\|\varphi\|^2$. Collecting terms, we obtain

$$\begin{aligned} J &\geq \frac{3}{N^2-1} [B_1 + NB_2 + 2N^2B_3] \|\varphi\|^2 \\ &\geq \frac{3}{N(N^2-1)} [B_1 + NB_2 + 2N^2B_3] \left(1 - \frac{15}{(N+1)(N+3)}\right)^{-1} \|f\|_2^2 . \end{aligned}$$

combining this estimate with (3.18) and (3.13), and simplifying the sums, we obtain the desired bound on $I_1 + I_2 + I_3 = \mathcal{G}_N(f, f)$. \square

4 Lower bound for Δ_N

We now show how to decompose an admissible trial function f in the variational formula for Δ_N into components g and h where $g \in \mathcal{A}_N$, and h satisfies $\langle h, P^{(\gamma)}h \rangle = 0$, which means that h makes no contribution to the negative term $\langle f, P^{(\gamma)}f \rangle$ in the induction bound from Theorem 2.2. We use this, and further correlation estimates, to extend our lower bound for $\widehat{\Delta}_N$ into one for Δ_N .

4.1 The trial function decomposition

Let Π denote the projection onto the space of functions orthogonal to the constants on $L^2(S^{N-1}(\sqrt{N}))$. Then the operator $\Pi P^{(\gamma)}\Pi$ is clearly self adjoint.

For any f orthogonal to the constants,

$$\langle f, P^{(\gamma)}f \rangle = \langle f, \Pi P^{(\gamma)}\Pi f \rangle . \quad (4.1)$$

now decompose f as $f = g + h$ where h is in the null space of $\Pi P^{(\gamma)}\Pi$, and g is in the range. Notice that f and g are orthogonal, so that

$$\|f\|_2^2 = \|g\|_2^2 + \|h\|_2^2 .$$

By the definition of h and (4.1),

$$\langle f, P^{(\gamma)}f \rangle = \langle g, P^{(\gamma)}g \rangle , \quad (4.2)$$

and hence

$$\int_{S^{N-1}(\sqrt{N})} W^{(\gamma)} f^2 d\sigma - \langle f, P^{(\gamma)}f \rangle_{L^2(S^{N-1}(\sqrt{N}))} = \int_{S^{N-1}(\sqrt{N})} W^{(\gamma)} f^2 d\sigma - \langle g, P^{(\gamma)}g \rangle_{L^2(S^{N-1}(\sqrt{N}))} . \quad (4.3)$$

Notice that h makes no contribution to the negative term on the right side of (4.3). In fact, an even stronger form of (4.2) is true, and will be useful to us:

4.1 LEMMA. *Let h be any function in $L^2((S^{N-1}(\sqrt{N})))$ that is orthogonal to the constants, and is in the null space of $\Pi P^{(\gamma)}\Pi$. Then for each k ,*

$$P_k h = 0 . \quad (4.4)$$

Proof. Since $\Pi h = h$, we have

$$0 = \langle h, \Pi P^{(\gamma)} \Pi h \rangle = \langle h, P^{(\gamma)} h \rangle = \frac{1}{N} \sum_{k=1}^N \int_{S^{N-1}(\sqrt{N})} \left(\frac{N - v_k^2}{N - 1} \right)^\gamma |P_k h|^2 d\sigma .$$

Since $\left(\frac{N - v_k^2}{N - 1} \right)^\gamma \geq 0$ almost everywhere, it must be the case that $|P_k h|^2$ vanishes identically. \square

The other key feature of the decomposition is that $g \in \mathcal{A}_N$; i.e., here is a function φ of a single variable such that $\varphi(v_k) \in L^2(S^{N-1}(\sqrt{N}))$ for each k (or equivalently, for any k), and

$$g(v) = \sum_{j=1}^N \varphi(v_j) . \quad (4.5)$$

That is, the range of $P^{(\gamma)}$ lies in the subspace \mathcal{A}_N of \mathcal{H}_N that figures in the definition (1.11). This is because as long as f is symmetric, so is Πf , and then $P^{(\gamma)} \Pi f$ has this form, and applying Π preserves this form. Here we are using symmetry to ensure that we need just one and the same function φ for each coordinate.

We now use the decomposition introduced at the beginning of this section to reduce the estimation of Δ_N to the estimation of $\hat{\Delta}_N$.

We return to

$$\mathcal{F}_N(f, f) = \frac{1}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} w^{(\gamma)}(v_k) [f - P_k f]^2 d\sigma \right] \quad (4.6)$$

and introduce the decomposition $f = g + h$. Then since $P_k h = 0$ for each k ,

$$\begin{aligned} [f - P_k f]^2 &= [g - P_k g]^2 + h^2 + 2[g - P_k g]h , \\ \mathcal{F}_N(f, f) &= \mathcal{F}_N(g, g) + \frac{2}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} w^{(\gamma)}(v_k) g h d\sigma \right] + \int_{S^{N-1}(\sqrt{N})} W^{(\gamma)}(v) h^2(v) d\sigma . \end{aligned} \quad (4.7)$$

We first estimate the cross terms.

4.2 LEMMA. *With g and h as above*

$$\left| \frac{2}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} w^{(\gamma)}(v_k) g h d\sigma \right] \right| \leq \frac{N-1}{N} \frac{C_N}{N^2} 2 \|g\|_2 \|h\|_2 \quad (4.8)$$

where C_N is given by (1.22).

Proof. We rewrite (4.8) as

$$\frac{2}{N} \sum_{k=1}^N \left[\int_{S^{N-1}(\sqrt{N})} \left[w^{(\gamma)}(v_k) - 1 - \gamma \frac{1 - v_k^2}{N - 1} \right] g h d\sigma \right]$$

since $\sum_{k=1}^N \left[1 - \gamma \frac{1 - v_k^2}{N - 1} \right] = N$, and g and h are orthogonal. Introducing $g = \sum_{j=1}^N \varphi(v_j)$, the quantity becomes

$$\frac{2}{N} \sum_{k=1}^N \sum_{j \neq k} \left[\int_{S^{N-1}(\sqrt{N})} \left[w^{(\gamma)}(v_k) - 1 - \gamma \frac{1 - v_k^2}{N - 1} \right] \varphi(v_j) h d\sigma \right] \quad (4.9)$$

where the term $j = k$ vanishes since $P_k h = 0$.

Next let $P_{\{j,k\}}$ denote the orthogonal projection onto the subspace of functions depending only on v_j and v_k . Evidently, we may replace h by $P_{\{j,k\}}h$ in each summand above. Then since

$$\left[w^{(\gamma)}(v_k) - 1 - \gamma \frac{1 - v_k^2}{N - 1} \right]^2 \leq \frac{(1 - \gamma)^2}{(N - 1)^4} (1 - v_k^2)^4 ,$$

the quantity in (4.9) is bounded in magnitude by

$$(1 - \gamma) \frac{1}{(N - 1)^2} \frac{2}{N} \sum_{k=1}^N \sum_{j \neq k} \int_{S^{N-1}(\sqrt{N})} |1 - v_k^2|^2 |\varphi(v_j)| |P_{\{j,k\}}h| d\sigma \quad (4.10)$$

By the Schwarz inequality, and then the definition of the K operator, with $\psi(v)$ denoting the function $(1 - v_k^2)^4$,

$$\int_{S^{N-1}(\sqrt{N})} |1 - v_k^2|^2 |\varphi(v_j)| |P_{\{j,k\}}h| d\sigma \leq \left(\int_{S^{N-1}(\sqrt{N})} K\psi(v_j) |\varphi(v_j)|^2 \right)^{1/2} \|P_{\{j,k\}}h\|_2 .$$

By the definition of the K operator and ψ , $K\psi(v)$ is a convex function of v^2 , and hence

$$0 \leq K\psi(v) \leq \max\{K\psi(0), K\psi(\sqrt{N})\} .$$

By direct computation

$$\begin{aligned} K\psi(v) = 1 - 4 \frac{N - v^2}{N - 1} + 18 \frac{(N - v^2)^2}{(N - 1)(N + 1)} \\ - 60 \frac{(N - v^2)^3}{(N - 1)(N + 1)(N + 3)} + 105 \frac{(N - v^2)^4}{(N - 1)(N + 1)(N + 3)(N + 5)} . \end{aligned}$$

Simple estimates show $K\psi(0) < K\psi(\sqrt{N}) \leq 60$ for all N . Therefore,

$$\left(\int_{S^{N-1}(\sqrt{N})} K\psi(v_j) |\varphi(v_j)|^2 \right)^{1/2} \leq \sqrt{60} \|\varphi\|_2 \|P_{\{j,k\}}h\|_2 .$$

Then, by symmetry, for each $j \neq k$, and Theorem 6.3,

$$\|P_{\{j,k\}}h\|_2^2 = \langle h, P_{\{j,k\}}h \rangle = \binom{N}{2}^{-1} \sum_{i < \ell} \langle h, P_{i,\ell}h \rangle \leq \left[\frac{2}{N - 1} + \frac{8N}{(N - 2)(N - 4)^2} \right] \|h\|_2^2 .$$

Further, Lemma 3.3 gives us

$$\|\varphi\|_2 \leq \left(1 - \frac{15}{(N + 1)(N + 3)} \right)^{-1/2} \frac{\|g\|_2}{\sqrt{N}} .$$

Combining these yields the result. \square

4.3 LEMMA. *With h as above*

$$\int_{S^{N-1}(\sqrt{N})} W^{(\gamma)}(v) h^2(v) d\sigma \geq \left(1 - \frac{1 - \gamma}{N - 1} \right) \|h\|_2^2 . \quad (4.11)$$

Proof. Simply use (2.26). \square

We come to:

Proof of Theorem 1.4. From (4.7), Theorem 2.2, Lemma 4.2 and Lemma 4.3,

$$\frac{N}{N-1} \mathcal{F}_N(f, f) \geq \left(1 - \frac{A_N}{N^2}\right) \|g\|_2^2 - \frac{C_N}{N^2} 2\|g\|_2 \|h\|_2 + \left(1 + \frac{\gamma N - 1}{(N-1)^2}\right) \|h\|_2^2. \quad (4.12)$$

Then using $2\|g\|_2 \|h\|_2 \leq \|f\|_2^2$, we obtain

$$\frac{N}{N-1} \mathcal{F}_N(f, f) \geq \left(1 - \frac{A_N + C_N}{N^2}\right) \|f\|_2^2, \quad (4.13)$$

from which the result follows. \square

Since $\lim_{N \rightarrow \infty} A_N =: A$ and $\lim_{N \rightarrow \infty} C_N =: C$ exist, for all $0 \leq \gamma \leq 1$, there exists an N_0 so that $N^2 > A_N + C_N$ for all $N \geq N_0$, and

$$\prod_{j \geq N_0}^{\infty} \left(1 - \frac{A_N + C_N}{N^2}\right) > 0.$$

By Theorem 2.8

$$\Delta_{N_0} \geq 4N^{\gamma-1} \left(\prod_{j=3}^{N_0} \left[1 - \frac{4j+1}{(j-1)^2(j+1)}\right] \right) > 0.$$

Altogether, we have

$$\liminf_{N \rightarrow \infty} \Delta_N \geq 4N^{\gamma-1} \left(\prod_{j=3}^{N_0} \left[1 - \frac{4j+1}{(j-1)^2(j+1)}\right] \right) \prod_{j \geq N_0}^{\infty} \left(1 - \frac{A_N + C_N}{N^2}\right) > 0,$$

which proves the Kac conjecture for $0 < \gamma < 1$.

4.2 The structure of the gap eigenfunction

The exact computation of Δ_N for $\gamma = 0$ [1] shows that in this case, $\Delta_N < \Delta_{N-1}$ for all N . It seems quite plausible that for *all* γ , Δ_N is monotone decreasing in N , but we have not been able to show this. All of our work so far in this paper has focused on lower bounds, for the obvious reasons.

Nonetheless, the conjectured monotonicity of Δ_N would have a significant consequence. Fix $\gamma > 0$. For any $N \geq 3$, let f_N be a normalized eigenfunction of L_N with $L_N f_N = -\Delta_N f_N$. Let $f_N = g_N + h_N$ be the trial function decomposition of f_N . Define α_N by

$$\alpha_N := \|h_N\|^2.$$

In passing from (4.12) to (4.13) we have simply discarded the term $\frac{\gamma N - 1}{(N-1)^2} \|h\|^2$. Taking $f = f_N$ and keeping this term, we deduce

$$\Delta_N \geq \left(1 - \frac{A_N + C_N}{N^2} + \alpha_N \frac{\gamma N - 1}{(N-1)^2}\right) \Delta_{N-1}.$$

If we then knew that $\Delta_N \leq \Delta_{N-1}$ for all N , we could conclude that

$$\alpha_N \leq \frac{A_N + C_N}{\gamma N}$$

for all N , and since $\lim_{N \rightarrow \infty} A_N = A$, and $\lim_{N \rightarrow \infty} C_N = C$ exist and are finite, this would mean that $\alpha_N = \mathcal{O}(1/N)$. Instead, we can prove:

4.4 PROPOSITION. *There is an infinite sequence of integers $\{N_k\}$ such that for each k , $\alpha_{N_k} \leq 1/\log(N_k)$.*

Proof. If this were not the case, then

$$\sum_{N=3}^{\infty} \left[-\frac{A_N + C_N}{N^2} + \alpha_N \frac{\gamma N - 1}{(N-1)^2} \right]$$

would diverge to $+\infty$, and this would imply that Δ_N would increase without bound. However, the trial function calculations in the next section show that this is not the case. \square

The next bound shows that if one could strengthen this to $\alpha_N = o(1/N^\gamma)$ for all large N , one would conclude that

$$\lim_{N \rightarrow \infty} (\widehat{\Delta}_N - \Delta_N) = 0 .$$

4.5 PROPOSITION. *Suppose that $\alpha_N = o(1/N^\gamma)$. There is a computable constant C so that for all sufficiently large N ,*

$$\Delta_N \geq \widehat{\Delta}_N(1 - \alpha_N) - C\sqrt{N^\gamma \alpha_N} .$$

Proof. Let f_N be a normalized gap eigenfunction for L_N , and let $f_N = g_N + h_N$ be its decomposition as above. Then

$$\Delta_N = \mathcal{E}_N(f_N, f_N) = \mathcal{E}_N(g_N, g_N) + 2\mathcal{E}_N(g_N, h_N) + \mathcal{E}_N(h_N, h_N) .$$

We note that the cross term cannot be positive since otherwise replacing f_N by $g_N - h_N$ would lower the value of $\mathcal{E}_N(f_N, f_N)$ while respecting the constraints. We shall discard the term $\mathcal{E}_N(h_N, h_N)$, and estimate the cross term from below. To do this, write $g(v) = \sum_{j=1}^N \varphi(v_j)$ as above, and define

$$g_{i,j} = (v_i^2 + v_j^2)^\gamma \frac{1}{2\pi} \int_{-\pi}^{\pi} [\varphi(v_i \cos \theta + v_j \sin \theta) + \varphi(-v_i \sin \theta - v_j \cos \theta) - \varphi(v_i) - \varphi(v_j)] d\theta ,$$

so that

$$L_N g_N = N \binom{N}{2}^{-1} \sum_{i < j} g_{i,j} .$$

We have

$$\begin{aligned} -\mathcal{E}_N(g_N, h_N) &= \langle L_N g_N, h_N \rangle \\ &= N \binom{N}{2}^{-1} \sum_{i < j} \langle g_{i,j}, h_N \rangle \\ &= N \binom{N}{2}^{-1} \sum_{i < j} \langle g_{i,j}, P_{\{i,j\}} h_N \rangle \\ &\leq N \left(\binom{N}{2}^{-1} \sum_{i < j} \|g_{i,j}\|^2 \right)^{1/2} \left(\binom{N}{2}^{-1} \sum_{i < j} \|P_{\{i,j\}} h_N\|^2 \right)^{1/2} \end{aligned}$$

By Theorem 6.3,

$$\binom{N}{2}^{-1} \sum_{i < j} \|P_{\{i,j\}}\|^2 \leq \frac{C}{N} \|h_N\|^2 = \frac{C}{N} \alpha_N .$$

Also, using the bound $(v_i^2 + v_j^2)^\gamma \leq N^\gamma$,

$$N \binom{N}{2}^{-1} \sum_{i < j} \|g_{i,j}\|^2 \leq N^\gamma \mathcal{E}_N(g_N, g_N) .$$

Altogether, we have

$$\Delta_N \geq \mathcal{E}_N(g_N, g_N) - 2\sqrt{CN^\gamma \alpha_N} \sqrt{\mathcal{E}_N(g_N, g_N)} .$$

Now, $\mathcal{E}_N(g_N, g_N) \geq \widehat{\Delta}_N \|g_N\|^2 = \widehat{\Delta}_N (1 - \alpha_N)$. Hence for all sufficiently large N , the difference above decreases if we replace $\mathcal{E}_N(g_N, g_N)$ by $\widehat{\Delta}_N (1 - \alpha_N)$. Renaming the constant, this yields the desired bound. \square

5 Spectral gap for the linearized collision operator

Our main goal in this section is to prove Theorem 1.5, which says that for $0 \leq \gamma < 1$,

$$\Lambda \geq \limsup_{N \rightarrow \infty} \widehat{\Delta}_N ,$$

where Λ is the spectral gap for the linearized Kac-Boltzmann equation.

In the proof, we shall use several lemmas. Let $\rho_N(v)$ be the probability density on \mathbb{R} defined by

$$\rho_N(v) dv = d\nu_{M,1} .$$

That is, ρ_N is the density of the one dimensional marginal of the uniform probability measure on $S^{N-1}(\sqrt{N})$. Writing (6.3) out explicitly, one has

$$\rho_N(v) dv = K_N \left(1 - \frac{v^2}{N}\right)_+^{(N-3)/2} \quad \text{where} \quad K_N = \frac{1}{\sqrt{N\pi}} \frac{\Gamma(N/2)}{\Gamma((N-1)/2)} . \quad (5.1)$$

By Stirlings formula,

$$K_N = \frac{1}{\sqrt{2\pi}} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) . \quad (5.2)$$

5.1 LEMMA. *There is a constant C independent of N such that*

$$\rho_N(v) \leq CM(v) \quad \text{for all } v \in \mathbb{R} , \quad (5.3)$$

where $M(v)$ is given by (1.27).

Proof. Using the elementary estimate $e^{-v^2} \geq \left(1 - \frac{v^2}{N}\right)_+^N$, valid for all v , we deduce

$$\rho_N(v) \leq K_N e^{-v^2/2} e^{3v^2/2N} = K_N \sqrt{2\pi} e^{3v^2/2N} M(v) . \quad (5.4)$$

Then since $v^2 \leq N$ for all v in the support of ρ_N , we have

$$\rho_N(v) \leq K_N \sqrt{2\pi} e^{3/2} M(v) .$$

The claim now follows from (5.2). In fact, we see that for all sufficiently large N , the constant $C = e^2$ would suffice. \square

In particular, it follows from Lemma 5.1 that $\rho_N(v)/M(v)$ is uniformly bounded on \mathbb{R} . We now show that this ratio converges to 1 uniformly on bounded intervals as N tends to infinity.

5.2 LEMMA. *For all $R > 0$,*

$$\lim_{N \rightarrow \infty} \left(\sup_{-R \leq v \leq R} \left| \frac{\rho_N(v)}{M(v)} - 1 \right| \right) = 0 . \quad (5.5)$$

Proof. From (5.4), one has the upper bound

$$\frac{\rho_N(v)}{M(v)} - 1 \leq K_N \sqrt{2\pi} e^{3R^2/2N} - 1$$

for all $v \in [-R, R]$. By (5.2), the right hand side is $\mathcal{O}(1/N)$.

For the lower bound, we start from the elementary estimate

$$\left(1 - \frac{v^2}{N} \right)^N \geq e^{-v^2 - v^4/N} \quad \text{whenever} \quad v^2 \leq \frac{N}{2} , \quad (5.6)$$

Thus,

$$\frac{\rho_N(v)}{M(v)} - 1 \geq K_N \sqrt{2\pi} e^{-R^4/2N} - 1 .$$

Once again, by (5.2), the right hand side is $\mathcal{O}(1/N)$. □

Notice the proof of Lemma 5.2 gives a little more than is stated: One could let R grow with N like N^α for any $\alpha < 1/4$.

To apply these lemmas, let φ be any unit vector in $L^2(\mathbb{R}, M(v)dv)$ that is orthogonal to both 1 and v^2 . Define a_N , b_N and φ_N by

$$a_N = \int_{\mathbb{R}} \varphi(v) \rho_N(v) dv \quad b_N = \left(\int_{\mathbb{R}} (v^2 - 1)^2 \rho_N(v) dv \right)^{-1/2} \int_{\mathbb{R}} \varphi(v) (v^2 - 1) \rho_N(v) dv ,$$

and

$$\varphi_N(v) = \varphi(v) - a_N - b_N \left(\int_{\mathbb{R}} (v^2 - 1)^2 \rho_N(v) dv \right)^{-1/2} (1 - v^2) .$$

Notice that since 1 and $v_1^2 - 1$ are orthogonal with respect to the uniform probability measure on $S^{N-1}(\sqrt{N})$, 1 and $v^2 - 1$ are orthogonal with respect to $d\nu_{N,1}(v) = \rho_N(v)dv$. Thus, for each j , $\varphi_N(v_j)$ is orthogonal to both 1 and v_j^2 with respect to the uniform probability measure on $S^{N-1}(\sqrt{N})$.

The next lemma concerns the trial function that we shall use to prove Theorem 1.5

5.3 LEMMA. *Define the function $f_N(v)$ by*

$$f_N(v) = \frac{1}{\sqrt{N}} \sum_{j=1}^N \varphi_N(v_j) . \quad (5.7)$$

Then for each N , $f \in L^2(S^{N-1}(\sqrt{N}))$ and is orthogonal to the constants. Moreover,

$$\lim_{N \rightarrow \infty} \int_{S^{N-1}(\sqrt{N})} f_N^2(v) d\sigma = 1 . \quad (5.8)$$

Proof. By Lemma 5.1, there is a finite constant so that $\rho_N(v) \leq CM(v)$, and thus

$$\int_{S^{N-1}(\sqrt{N})} \varphi^2(v_1) d\sigma = \int_{\mathbb{R}} \varphi^2(v) \rho_N(v) dv \leq C \int_{\mathbb{R}} \varphi^2(v) M(v) dv \leq C .$$

This proves the square integrability, and then the orthogonality statement follows by construction.

Next, by Lemma 3.3,

$$\left(1 - \frac{15}{(N+1)(N+3)}\right) \int_{S^{N-1}(\sqrt{N})} \varphi_N^2(v_1) d\sigma \leq \int_{S^{N-1}(\sqrt{N})} f_N^2(v) d\sigma \leq \left(1 + \frac{3}{N+1}\right) \int_{S^{N-1}(\sqrt{N})} \varphi_N^2(v_1) d\sigma . \quad (5.9)$$

Also, by construction,

$$\int_{S^{N-1}(\sqrt{N})} \varphi_N^2(v_1) d\sigma = \int_{\mathbb{R}} \varphi^2(v) \rho_N(v) dv - a_N^2 - b_N^2 .$$

Again by Lemma 5.1, $\rho_N(v) \leq CM(v)$ for all v , and then since $\lim_{N \rightarrow \infty} \rho_N(v) = M(v)$ for all v , the Dominated Convergence Theorem yields

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} \varphi^2(v) \rho_N(v) dv = \int_{\mathbb{R}} \varphi^2(v) M(v) dv = 1 .$$

□

Proof of Theorem 1.5. By symmetry and direct computation,

$$\mathcal{E}_{N,E}(f_N, f_N) = N \int_{S^{N-1}(\sqrt{N})} (v_1^2 + v_2^2)^\gamma f_N(v) [f_N(v) - f_N^{1,2}(v)] d\sigma ,$$

and also

$$f_N(v) - f_N^{1,2}(v) = \frac{1}{\sqrt{N}} \left(\varphi_N(v_1) + \varphi_N(v_2) - 2\varphi_N^{(1,2)}(v_1, v_2) \right)$$

where

$$\varphi_N^{(1,2)}(v_1, v_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_N(\cos \theta v_1 + \sin \theta v_2) d\theta .$$

Thus,

$$\begin{aligned} \mathcal{E}_{N,E}(f_N, f_N) &= \int_{S^{N-1}(\sqrt{N})} (v_1^2 + v_2^2)^\gamma (\varphi_N(v_1) + \varphi_N(v_2)) \left(\varphi_N(v_1) + \varphi_N(v_2) - 2\varphi_N^{(1,2)}(v_1, v_2) \right) d\sigma \\ &+ \sum_{j=3}^N \varphi_N(v_j) \int_{S^{N-1}(\sqrt{N})} (v_1^2 + v_2^2)^\gamma \left(\varphi_N(v_1) + \varphi_N(v_2) - 2\varphi_N^{(1,2)}(v_1, v_2) \right) d\sigma . \end{aligned} \quad (5.10)$$

Using our results on the spectrum of $K_{(2)}$, the orthogonality properties of φ_N , and the trivial bound $(v_1^2 + v_2^2)^\gamma \leq N^\gamma$, we have that

$$\left| \sum_{j=3}^N \varphi_N(v_j) \int_{S^{N-1}(\sqrt{N})} (v_1^2 + v_2^2)^\gamma \left(\varphi_N(v_1) + \varphi_N(v_2) - 2\varphi_N^{(1,2)}(v_1, v_2) \right) d\sigma \right| \leq \frac{C}{N^{1-\gamma}} .$$

Next, simple computations show that for each N ,

$$\left(\varphi_N(v_1) + \varphi_N(v_2) - 2\varphi_N^{(1,2)}(v_1, v_2) \right) = \left(\varphi(v_1) + \varphi(v_2) - 2\varphi^{(1,2)}(v_1, v_2) \right) ,$$

and then that the value of

$$\int_{S^{N-1}(\sqrt{N})} (v_1^2 + v_2^2)^\gamma (\varphi_N(v_1) + \varphi_N(v_2)) \left(\varphi_N(v_1) + \varphi_N(v_2) - 2\varphi_N^{(1,2)}(v_1, v_2) \right)$$

is unchanged upon replacing φ_N by φ . Finally, a simple dominated convergence argument based on the two dimension analog of $\rho_n(v) \leq CM(v)$ shows that

$$\lim_{N \rightarrow \infty} \mathcal{E}_{N,E}(f_N, f_N) = \langle \varphi, \mathcal{L}\varphi \rangle_{L^2(\mathbb{R}, M(v)dv)} .$$

Combining this with Lemma 5.3, we have

$$\langle \varphi, \mathcal{L}\varphi \rangle_{L^2(\mathbb{R}, M(v)dv)} = \lim_{N \rightarrow \infty} \frac{\mathcal{E}_{N,E}(f_N, f_N)}{\|f_N\|_2} \geq \limsup_{N \rightarrow \infty} \hat{\Delta}_N .$$

As φ is an arbitrary trial function in the variational definition of Λ , this proves Theorem 1.5. \square

6 Correlation operators on the sphere

In this section we work on $S^{N-1} = S^{N-1}(1)$, the unit sphere in \mathbb{R}^N , and we let $d\sigma_N$ denote the uniform probability on S^{N-1} . The relation between the uniform probability measure on $S^{N-1}(\sqrt{N})$ that was studied in the last section suggests that the coordinate functions on $(S^{N-1}, d\sigma_N)$, regarded as a probability space, should be “nearly independent” for large N , at least if one does not “look at too many of them at once”. Our goal in this section is to prove results that precisely express, in a quantified manner, these assertions. Using the unitary rescaling operation (1.13) we shall then be able to apply our results in $L^2(S^{N-1}(\sqrt{N}))$. However, the derivation of the results shall be easier if we work on the unit sphere.

We begin by introducing some notation. Let \mathcal{H} denote the Hilbert space $L^2(S^{N-1}, d\sigma_N)$. Given any non-empty subset $A \subset \{1, \dots, N\}$, let \mathcal{H}_A denote the subspace of \mathcal{H} that is the closure of the set of all polynomials in the variables v_j with $j \in A$. That is, loosely speaking, \mathcal{H}_A is the subspace of functions only depending on coordinates with indices in A . Let P_A denote the orthogonal projection of \mathcal{H} onto \mathcal{H}_A . In the special case that $A = \{j\}$, we simply write P_j to denote this projection. This usage is consistent with our previous use of the notation P_j .

For each $m = 1, \dots, N-1$, let $\pi_{\{1, \dots, m\}}$ denote the map

$$\pi_{\{1, \dots, m\}}(v) = (v_1, \dots, v_m) . \quad (6.1)$$

For simplicity, we write π_j to mean $\pi_{\{j\}}$. The image of S^{N-1} under $\pi_{\{1, \dots, m\}}$ is the unit ball B^m in \mathbb{R}^m , and by a standard computation, for any continuous function ψ on B^m ,

$$\int_{S^{N-1}} \psi[\pi_{\{1, \dots, m\}}(v)] d\sigma = \int_{B^m} \psi[w] d\nu_{N,m}(w) , \quad (6.2)$$

where

$$d\nu_{N,m}(w) = \frac{|S^{N-m-1}|}{|S^{N-1}|} (1 - |w|^2)^{(N-m-2)/2} dw \quad (6.3)$$

where $|S^{k-1}|$ denote the surface area of S^{k-1} in \mathbb{R}^k . We define $\mathcal{K}_{N,m}$ to be the Hilbert space $L^2(B^m, \nu_{N,m})$.

Now define $\phi_{N,m} : S^{N-m-1} \times B^m \rightarrow S^{N-1}$ by

$$\phi_{N,m}(y, w) = (\sqrt{1 - |w|^2}y_1, \dots, \sqrt{1 - |w|^2}y_{N-m}, w_1, \dots, w_m) . \quad (6.4)$$

Then for any continuous function h on S^{N-1} ,

$$\int_{S^{N-1}} h[x] d\sigma(x) = \int_{B^m} \left[\int_{S^{N-m-1}} h[\phi_{N,m}(y, w)] d\sigma(y) \right] d\nu_{N,m}(w) . \quad (6.5)$$

For any $m \leq N/2$, we define the m -particle correlation operator $K_{(N,m)}$ as a self adjoint operator on $\mathcal{K}_{N,m}$ through

$$\langle K_{(N,m)} f, g \rangle_{\mathcal{K}_{N,m}} = \int_{S^{N-1}} f(v_1, \dots, v_m) g(v_{N-m+1}, \dots, v_N) d\sigma . \quad (6.6)$$

Then using (6.4), we find

$$K_{(N,m)} f(w_1, \dots, w_m) = \int_{S^{N-m-1}} f(\sqrt{1 - |w|^2}y_1, \dots, \sqrt{1 - |w|^2}y_m) d\nu_{N-m,m}(y) . \quad (6.7)$$

6.1 THEOREM. For all $m \leq N$ and all $\mathbb{N} \geq 3$, the non zero eigenvalues of $K_{(N,m)}$ are given by

$$\kappa_{N,m}(k) = (-1)^k \frac{\Gamma\left(\frac{N-m}{2}\right)}{\Gamma\left(\frac{N-m}{2} + k\right)} \frac{\Gamma\left(\frac{m}{2} + k\right)}{\Gamma\left(\frac{m}{2}\right)} \quad \text{for} \quad k = 0, 1, 2, \dots . \quad (6.8)$$

For each $k \geq 0$, one has

$$|\kappa_{N,m}(k)| > |\kappa_{N,m}(k+1)| . \quad (6.9)$$

Moreover, for each k , the eigenspace corresponding to $\kappa_{N,m}(k)$ is one dimensional, and is spanned by a polynomial of degree k in $|y|^2$, $y \in B^m$. In particular, the eigenspace corresponding to $\kappa_{N,m}(1)$ is spanned by

$$|y|^2 - m/N . \quad (6.10)$$

Proof. Observe from (6.7) that $K_{(N,m)} f$ is always radial. Moreover, by the symmetries of $d\nu_{N-m,m}(y)$, if f is a polynomial of total degree k in y_1, \dots, y_m , so is $K_{(N,m)} f$. Thus, the eigenfunctions of $K_{(N,m)}$ that are not in the null space of $K_{(N,m)}$ are polynomials in $|y|^2$, where $y \in B^m$ and hence $0 \leq |y|^2 \leq 1$. The polynomials are thus identified as (shifted and scaled) Jacobi polynomials. But even more easily, it is easy to see that the eigenvalue $\kappa_{N,m}(k)$ of $K_{(N,m)}$ that corresponds to the eigenfunction that is a polynomial of order $|v|^{2k}$ is given by

$$\kappa_{N,m}(k) = (-1)^k \int_{B^m} |y|^{2k} d\nu_{N-m,m}(y) . \quad (6.11)$$

This integral can easily be evaluated in terms of the Beta function, $B(x, y)$. Using (6.3), we have

$$\begin{aligned} \int_{B^m} |y|^{2k} d\nu_{N-m,m}(y) &= \frac{|S^{N-2m-1}| |S^{m-1}|}{|S^{N-1}|} \int_0^1 (1-r^2)^{(N-2m-2)/2} (r^2)^{(m-1+2k)/2} dr \\ &= \frac{|S^{N-2m-1}| |S^{m-1}|}{2|S^{N-1}|} \int_0^1 (1-x)^{(N-2m-2)/2} (x)^{(m+2k-2)/2} dx \\ &= \frac{|S^{N-2m-1}| |S^{m-1}|}{2|S^{N-1}|} B\left(\frac{N}{2} - m, \frac{m}{2} + k\right) . \end{aligned} \quad (6.12)$$

Since the right hand side equals one for $k = 0$, it follows that

$$\frac{|S^{N-2m-1}||S^{m-1}|}{2|S^{N-1}|} = \left[B\left(\frac{N}{2} - m, \frac{m}{2}\right) \right]^{-1}.$$

This leads to the explicit formula

$$|\kappa_{N,m}(k)| = \frac{B(N/2 - m, m/2 + k)}{B(N/2 - m, m/2)}.$$

Finally, using the identity $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, one arrives at (6.8).

While (6.9) can be deduced from (6.8), it is much simpler to observe that since $|y|^{2k} > |y|^{2k+2}$ almost everywhere on B^m , (6.9) follows directly from (6.11). In particular, there is no degeneracy in the non-zero spectrum, and each eigenvalue $\kappa_{N,m}(k)$ has a one dimensional eigenspace spanned by a polynomial of degree k in $|y|^2$. Since functions in the eigenspace for $\kappa_{N,m}(1)$ must be orthogonal to the constants, it follows that this eigenspace is spanned by the function given in (6.10). \square

Using (6.10) and the identity $\Gamma(x+1) = x\Gamma(x)$, one readily computes

$$\kappa_{N,m}(1) = -\frac{m}{N-m} \quad (6.13)$$

and

$$\kappa_{N,m}(2) = \frac{m(m+2)}{(N-m)(N-m+2)}. \quad (6.14)$$

We now relate these results to the idea that the coordinate functions on $(S^{N-1}, d\sigma_N)$, regarded as a probability space, should be “nearly independent” for large N , at least if one does not “look at too many of them at once”.

Fix any $m \leq N/2$, and let A and B be two *disjoint* subsets of $\{1, \dots, N\}$, each of cardinality m . Let f and g be any two functions in $L^2(B^m, d\nu_{N,m})$ that are orthogonal to the constants. Then, by the definitions of $d\nu_{N,m}$, $K_{(N,m)}$ and symmetry,

$$\int_{S^{N-1}} (f \circ \pi_A) (g \circ \pi_B) d\sigma_N = \int_{B^m} f(y) [K_{(N,m)}g](y) d\nu_{N,m}(y). \quad (6.15)$$

If the coordinate functions were independent on $(S^{N-1}, d\sigma_N)$, then the left hand side of (6.9) would equal

$$\int_{S^{N-1}} (f \circ \pi_A) d\sigma_N \int_{S^{N-1}} (g \circ \pi_B) d\sigma_N = \int_{B^m} f(y) d\nu_{N,m}(y) \int_{B^m} g(y) d\nu_{N,m}(y) = 0.$$

That is, the random variables $(f \circ \pi_A)$ and $(g \circ \pi_B)$ would be uncorrelated. However, the coordinate functions are not independent on $(S^{N-1}, d\sigma_N)$, and the size of the right hand side of (6.9) measures the resulting correlations between $(f \circ \pi_A)$ and $(g \circ \pi_B)$. By Theorem 6.1 and the computation (6.13), we have

$$\left| \int_{B^m} f(y) [K_{(N,m)}g](y) d\nu_{N,m}(y) \right| \leq \frac{m}{N-m} \|f\|_{L^2(B^m, d\nu_{N,m})} \|g\|_{L^2(B^m, d\nu_{N,m})}.$$

That is,

$$\left| \int_{S^{N-1}} (f \circ \pi_A) (g \circ \pi_B) d\sigma_N \right| \leq \frac{m}{N-m} \left(\int_{S^{N-1}} (f \circ \pi_A)^2 d\sigma_N \right)^{1/2} \left(\int_{S^{N-1}} (g \circ \pi_B)^2 d\sigma_N \right)^{1/2}. \quad (6.16)$$

One sees from (6.16) that if m is small compared to N , then $(f \circ \pi_A)$ and $(g \circ \pi_B)$ have only a small correlation. However, if we look at too many variables at once, so that m is not small compared with N , there can be significant correlation. For example, if N is even and $m = N/2$, then $m/(N-m) = 1$, and $(f \circ \pi_A)$ and $(g \circ \pi_B)$ can be *fully* correlated. Indeed, if $N = 2m$ and one knows the value of $v_1^2 + \cdots + v_m^2$, then knows the value of $v_{m+1}^2 + \cdots + v_N^2$ with complete certainty.

Let us now consider the cases $m = 1$ and $m = 2$ with N large, so that correlations will be small. Let f be any function in \mathcal{H} that is orthogonal to the constants. Then for each $k = 1, \dots, N$, $P_k f$ is also orthogonal to the constants, and for $j \neq k$, $P_j f$ and $P_k f$ would be nearly orthogonal. If they were *exactly* orthogonal, Bessel's inequality would imply that

$$\sum_{k=1}^N \|P_k f\|_{\mathcal{H}}^2$$

would be no larger than $\|f\|_{\mathcal{H}}^2$. It turns out that for $m = 1$ and large N , the correlations are small enough that something almost as good as this is true:

6.2 THEOREM. *Let f be any function in \mathcal{H} that is orthogonal to the constants. Then*

$$\frac{1}{N} \sum_{k=1}^N \|P_k f\|_{\mathcal{H}}^2 \leq \left(\frac{1}{N} + \frac{3}{N(N+1)} \right) \|f\|_{\mathcal{H}}^2.$$

Theorem 6.2 is proved in [2]. The following theorem is a companion to it for $m = 2$. For f orthogonal to the constants and N large, $P_{\{1,2\}} f$, $P_{\{3,4\}} f$, $P_{\{5,6\}} f$, and so forth, should be nearly orthogonal and so one might expect that

$$\sum_{\{j : 2 \leq 2j \leq N\}} \|P_{\{2j-1, 2j\}} f\|_{\mathcal{H}}^2$$

should not be much larger than $\|f\|_{\mathcal{H}}^2$. Since there are essentially $N/2$ terms in the sum, and the measure $d\sigma_N$ is permutation invariant, one might expect a result of the following type:

6.3 THEOREM. *For all $N \geq 3$ and all f orthogonal to the constants,*

$$\binom{N}{2}^{-1} \sum_{i < j} \|P_{\{i,j\}} f\|^2 \leq \left[\frac{2}{N-1} + \frac{8N}{(N-2)(N-4)^2} \right] \|f\|^2.$$

Proof. Let M denote the integer part of $N/2$. Let \mathcal{A} denote any set of M *disjoint* pairs of indices in $\{1, \dots, N\}$. Let $P_{\mathcal{A}}$ denote the self-adjoint operator

$$P_{\mathcal{A}} = \frac{1}{M} \sum_{\alpha \in \mathcal{A}} P_{\alpha}.$$

We wish to bound

$$\sup \{ \langle f, P_{\mathcal{A}} f \rangle : \|f\|_2 = 1, \langle f, 1 \rangle = 0 \text{ and } f \text{ is symmetric} \} \quad (6.17)$$

Define the function

$$\eta(v_1, v_2) = \frac{2}{N} - v_1^2 - v_2^2$$

on the unit ball in \mathbb{R}^2 . This function spans the eigenspace of $K_{(N,2)}$ with eigenvalue $\kappa_{N,2} = -2/(N-2)$. For the pair $\gamma = (\gamma_1, \gamma_2)$, define

$$\xi_\gamma(v) = \eta(v_{\gamma_1}, v_{\gamma_2}) ,$$

Then since the average of the ξ_γ over all pairs is zero, for f is symmetric, f is orthogonal to each ξ_γ . In particular, such an f is orthogonal to each ξ_α with $\alpha \in \mathcal{A}$. Since the pairs in \mathcal{A} are disjoint, and since η is an eigenfunction of $K_{(N,2)}$, it is evident that the span of $\{\xi_\alpha : \alpha \in \mathcal{A}\}$ is an invariant subspace of $P_{\mathcal{A}}$.

Let us say that a function f is \mathcal{A} -symmetric in case it is symmetric under interchanges of pairs of coordinates in \mathcal{A} . The space of \mathcal{A} -symmetric functions is evidently an invariant subspace of $P_{\mathcal{A}}$, and any (fully) symmetric function belongs to this space.

Altogether, the supremum in (6.17) is no larger than the largest eigenvalue of $P_{\mathcal{A}}$ on the subspace of \mathcal{A} -symmetric functions that are orthogonal to each ξ_α , $\alpha \in \mathcal{A}$.

Therefore, let f be any eigenfunction of $P_{\mathcal{A}}$ on this space for which the eigenvalue is non-zero. Since f is in the range of $P_{\mathcal{A}}$, it is necessarily of the form

$$f(v) = \sum_{\alpha \in \mathcal{A}} \varphi(\pi_\alpha(v))$$

for some function φ on the ball such that φ is orthogonal to η on the ball.

For such an f , we compute

$$\begin{aligned} \langle f, P_{\mathcal{A}} f \rangle &= \frac{1}{M} \sum_{\alpha, \beta, \gamma \in \mathcal{A}} \langle \varphi \circ \pi_\beta, P_\alpha \varphi \circ \pi_\gamma \rangle_{\mathcal{H}} \\ &= \frac{1}{M} \sum_{\alpha, \beta \in \mathcal{A}} \langle \varphi \circ \pi_\beta, \varphi \circ \pi_\alpha \rangle_{\mathcal{H}} + \frac{M-1}{M} \sum_{\alpha, \beta \in \mathcal{A}} \langle \varphi \circ \pi_\beta, [K_{N,2} \varphi] \circ \pi_\alpha \rangle_{\mathcal{H}} \\ &= \frac{1}{M} \|f\|^2 + \frac{(M-1)^2}{M} \langle \varphi + (M-1) [K_{N,2} \varphi], [K_{N,2} \varphi] \rangle_{\mathcal{K}_{N,2}} \end{aligned} \quad (6.18)$$

$$\langle f, P_{\mathcal{A}} f \rangle \leq \frac{1}{M} \|f\|^2 + \left(\frac{8}{N(N-2)} \right) \left(\frac{M-1}{M} \right)^2 \left(1 + (M-1) \frac{8}{N(N-2)} \right) M \|\varphi\|^2 .$$

Finally, we note that, again using the spectral properties of $K_{(N,2)}$, that

$$\|f\|^2 \geq M \|\varphi\|^2 \left(1 - (M-1) \frac{8}{N(N-2)} \right) .$$

Then, since for $0 < x < 1$, $(1+x) < (1-x)^{-1}$, we obtain

$$\langle f, P_{\mathcal{A}} f \rangle \leq \left[\frac{1}{M} + \left(\frac{8}{N(N-2)} \right) \left(\frac{M-1}{M} \right)^2 \left(1 - (M-1) \frac{8}{N(N-2)} \right)^{-2} \right] \|f\|^2 .$$

Dropping the factor $((M-1)/M)^2$ and replacing M by either $(N-1)/2$ or $N/2$ as appropriate, we reduce this to

$$\langle f, P_{\mathcal{A}} f \rangle \leq \left[\frac{2}{N-1} + \left(\frac{8}{N(N-2)} \right) \left(1 - \frac{4}{N} \right)^{-2} \right] \|f\|^2 .$$

Further simplifying this, and averaging over all choices of \mathcal{A} , we obtain the bound claimed in the theorem. \square

We close this section by carrying out the simple computations based on (6.7) that have been invoked in the proof of Lemma 3.6. First consider the case $\psi(w_1, w_2) = w_1^2$. Then

$$K_{(N,2)}\psi(w_1, w_2) = (1-|w|^2) \int_{B_2} |y_1|^2 d\nu_{N-2,2}(y) = (1-|w|^2) \int_{B_1} |y_1|^2 d\nu_{N-2,1}(y) = \frac{1}{N-3}(1-|w|^2) . \quad (6.19)$$

Likewise, for $\psi(w_1, w_2) = w_1^4$,

$$K_{(N,2)}\psi(w_1, w_2) = (1-|w|^2)^2 \int_{B_2} |y_1|^4 d\nu_{N-2,2}(y) = (1-|w|^2)^2 \int_{B_1} |y_1|^4 d\nu_{N-2,1}(y) = \frac{3}{(N-3)(N-1)}(1-|w|^2)^2 . \quad (6.20)$$

The next lemma records these computations in a form that is directly applicable to our problem here.

6.4 LEMMA. *Consider v_k^2 and v_k^4 as functions on $S^{N-1}(\sqrt{N})$. Then for k, j, ℓ all distinct,*

$$P_{\{j,\ell\}}v_k^2 = \frac{1}{N-3}[N - (v_j^2 + v_\ell^2)] \quad (6.21)$$

and

$$P_{\{j,\ell\}}v_k^4 = \frac{3}{(N-3)(N-1)}[N - (v_j^2 + v_\ell^2)]^2 . \quad (6.22)$$

Proof. First define w by $v = \sqrt{N}w$ so that w lies in the unit sphere. Then

$$P_{\{j,\ell\}}v_k^2 = NP_{\{j,\ell\}}w_k^2 = N \frac{1}{N-3}(1 - (w_j^2 + w_\ell^2)) = \frac{1}{N-3}(N - (v_j^2 + v_\ell^2)) ,$$

where we have used (6.19). One proves (6.22) by making analogous use of (6.20). \square

7 An evaluation formula for certain infinite products

Let $P(x)$ and $Q(x)$ be two polynomials. We are interested in calculating the infinite product

$$\lim_{N \rightarrow \infty} \prod_{j=M}^N \frac{P(j)}{Q(j)} ,$$

and determining when the limit is non-zero.

A necessary condition that the limit exists and is non-zero is that both polynomials are of the same degree K and that the coefficient of the two leading order terms are the same. This follows from the fact that in order for the limit to exist and to be nonzero the factors must be of the form $1 + R(j)$ where $R(j) = O(\frac{1}{j^2})$. Hence we may assume that

$$P(x) = (x - \mu_1) \cdots (x - \mu_K), \quad Q(x) = (x - \nu_1) \cdots (x - \nu_K),$$

where $\sum \mu_n = \sum \nu_n$. And if we seek an non-zero limit, another obvious requirement is that none of the polynomials vanish for any $j \geq M$ since then one of the factors is either not defined or vanishes.

7.1 THEOREM. *Under the assumption stated above,*

$$\lim_{N \rightarrow \infty} \Pi_{j=M}^N \frac{P(j)}{Q(j)} = \Pi_{n=1}^K \frac{\Gamma(M - \mu_n)}{\Gamma(M - \nu_n)}.$$

Proof. We write

$$\Pi_{j=M}^N (j - \mu) = \frac{\Gamma(N + 1 - \mu)}{\Gamma(M - \mu)}$$

and find

$$\Pi_{j=M}^N \frac{P(j)}{Q(j)} = \Pi_{n=1}^K \frac{\Gamma(N + 1 - \mu_n) \Gamma(M - \nu_n)}{\Gamma(M - \mu_n) \Gamma(N + 1 - \nu_n)}.$$

Using Stirling's formula

$$\Gamma(x) \approx \sqrt{2\pi x} x^{x-1/2} e^{-x}, \quad x \rightarrow \infty$$

we can write for N large

$$\Pi_{j=M}^N \frac{P(j)}{Q(j)} \approx \Pi_{n=1}^K \frac{(N + 1 - \mu_n)^{N+1/2-\mu_n} e^{-(N+1-\mu_n)} \Gamma(M - \nu_n)}{\Gamma(M - \mu_n) (N + 1 - \nu_n)^{N+1/2-\nu_n} e^{-(N+1-\nu_n)}}$$

which simplifies to

$$= \Pi_{n=1}^K \frac{(N + 1)^{-\mu_n} (1 - \frac{\mu_n}{N+1})^{N+1/2-\mu_n} e^{\mu_n} \Gamma(M - \mu_n)}{(N + 1)^{-\nu_n} (1 - \frac{\nu_n}{N+1})^{N+1/2-\nu_n} e^{\nu_n} \Gamma(M - \nu_n)}.$$

Using the assumption that $\sum_{n=1}^K \mu_n = \sum_{n=1}^K \nu_n$, we find for large N

$$\Pi_{j=M}^N \frac{P(j)}{Q(j)} = \Pi_{n=1}^K \frac{(1 - \frac{\mu_n}{N+1})^{N+1/2-\mu_n} e^{\mu_n} \Gamma(M - \mu_n)}{(1 - \frac{\nu_n}{N+1})^{N+1/2-\nu_n} e^{\nu_n} \Gamma(M - \nu_n)},$$

which converges to

$$\Pi_{n=1}^K \frac{\Gamma(M - \mu_n)}{\Gamma(M - \nu_n)}$$

□

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